Fuzzy Bimatrix Games with Single and Multiple Objective : introduction to the computational techniques

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Abstract: This paper introduces to the computational techniques of non-cooperative bimatrix games in an uncertain environment. Both single and multiple objective fuzzy-valued bimatrix games are considered theoretically with one numerical example. The presentation is centered on the Nishizaki and Sakawa models. These models are based on the maxmin and minmax principles of the classical matrix game theory. Equivalent nonlinear (possibly quadratic) programming problems are giving optimal solutions. The equilibrium solutions correspond to players maximizing a degree of attainment of the fuzzy goals. Besides the Nash equilibrium, the concept of α-Nash equilibrium supposes Nature be the third Player. The aggregation of all the fuzzy sets in the multiobjective models use the fuzzy decision rule by Bellman and Zadeh. This 'aggregation by a minimum component' consists in the intersection of the fuzzy sets, the fuzzy expected payoffs and the fuzzy goals. Numerical examples of two-players nonzero sum games are solved using the *MATHEMATICA 7.0.1* software. The numerical solutions are possibly local by using iterative methods.

Key–Words: Fuzzy bimatrix game, Single-objective, Multiobjective, Degree of attainment of a fuzzy goal, Quadratic programming

1 Introduction

The non-cooperative bimatrix games using fuzzy logic (Nguyen and Walker [15]) differ according two main aspects in the literature: the number of the objectives and the type of fuzziness of the goals and payoffs. Indeed, the bimatrix games may be with a single objective or with multiple objectives (, Chen [6], Chen and Larbani [7], Keller [9, 10],Nishizaki and Sakawa [16]). The uncertainty may also concern the goals or the payoffs (Bector and Chandra [1], Han *et al.* [8], Larbani [11], Wang *et al.* [24]) or both goals and payoffs (Nishizaki and Sakawa [16, 17], Vodyottama *et al.* [22]). This introduction presents a crisp bimatrix game with a single objective.

1.1 Single objective bimatrix game

Two Players I and II have mixed strategies given by the n-dimensional vector $\mathbf{x}$ and the m-dimensional vector $\mathbf{y}$, respectively. Mixed strategies of Players I and II are represented by probability distributions to their pure strategies. Let $\mathbf{e}_n$ be an n-dimensional vector of ones, $\mathbf{e}_m$ having a dimension m. Suppose that the strategy spaces of Player I and II are defined by the convex polytopes

$$S^m = \{\mathbf{x} \in \mathbb{R}^m_+,\ \mathbf{e}'_m\mathbf{x} = 1\}$$

and

$$S^n = \{\mathbf{y} \in \mathbb{R}^n_+,\ \mathbf{e}'_n\mathbf{y} = 1\},$$

respectively. The payoffs of Players I and II are the $m\times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ with real entries, respectively. The objectives of Players I and II are defined by the programming problems, respectively

$$\{\max_{\mathbf{x}}\ \mathbf{x}'\mathbf{A}\mathbf{y} \text{ subject to } \mathbf{e}'_m\mathbf{x} = 1,\ \mathbf{x} \geq 0\}$$

and

$$\{\max_{\mathbf{y}}\ \mathbf{x}'\mathbf{B}\mathbf{y} \text{ subject to } \mathbf{e}'_n\mathbf{y} = 1,\ \mathbf{y} \geq 0\},$$

respectively. The payoff domains for Players I and II are

$$D_1 = \{\mathbf{x}'\mathbf{A}\mathbf{y} |\ \mathbf{x} \in S^m,\ \mathbf{y} \in S^n\} \subseteq \mathbb{R}$$

and

$$D_2 = \{\mathbf{x}'\mathbf{B}\mathbf{y} |\ \mathbf{x} \in S^m,\ \mathbf{y} \in S^n\} \subseteq \mathbb{R},$$

respectively. Playing safe, the two players will select the strategies for which the maximum losses are minimum.

1.2 Equilibrium solution

Definition 1 *A **Nash equilibrium point** is a pair of mixed strategies* $(\boldsymbol{x}^*, \boldsymbol{y}^*)$*, such that the objectives of the two players are full filled simultaneously. We have*

$$\begin{aligned} \boldsymbol{x}'^* \boldsymbol{A} \boldsymbol{y}^* &= \max_{\boldsymbol{x}} \{\boldsymbol{x}' \boldsymbol{A} \boldsymbol{y}^* | \, \boldsymbol{e}'_m \boldsymbol{x} = 1, \, \boldsymbol{x} \geq 0\}, \\ \boldsymbol{x}'^* \boldsymbol{B} \boldsymbol{y}^* &= \max_{\boldsymbol{y}} \{\boldsymbol{x}'^* \boldsymbol{B} \boldsymbol{y} | \, \boldsymbol{e}'_n \boldsymbol{y} = 1, \, \boldsymbol{y} \geq 0\}. \end{aligned}$$

The value of the game is obtained at the point $(\boldsymbol{x}'^* \boldsymbol{A} \boldsymbol{y}^*, \boldsymbol{x}'^* \boldsymbol{B} \boldsymbol{y}^*)$.

Applying the Kuhn-Tucker necessary and sufficient conditions (see appendix A), we have the Equivalence Theorem 2.

Theorem 2 (Mangasarian and Stone [13]) *Let* $G = (S^m, S^n, \boldsymbol{A}, \boldsymbol{B})$ *be a bimatrix game, a necessary and sufficient condition that* $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ *be an equilibrium point is the solution of the quadratic programming (QP) problem*

$$\left. \begin{array}{rr} \max_{\boldsymbol{x},\boldsymbol{y},p,q} & \boldsymbol{x}'(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{y} - p - q \\ & \textit{subject to} \\ & \boldsymbol{B}'\boldsymbol{x} \leq q\boldsymbol{e}_n, \\ & \boldsymbol{A}\boldsymbol{y} \leq p\boldsymbol{e}_m, \\ & \boldsymbol{e}'_m \boldsymbol{x} = 1, \\ & \boldsymbol{e}'_n \boldsymbol{y} = 1, \\ & \boldsymbol{x} \geq 0, \, \boldsymbol{y} \geq 0, \end{array} \right]$$

where $p, q \in \mathbb{R}$ *are the negative of the multipliers associated with the constraints.*

Proof: see appendix B. ∎

The Lemke-Howson 's pivot algorithm [12] (see also, Milchtaich [14], von Stengel [23]) can be used for computing the equilibrium solutions [1].

1.3 Bimatrix game with fuzzy goals

Definition 3 *A **fuzzy goal** for Player I is a fuzzy set* $\tilde{G}_1$ *represented by the membership function (MF)* $\mu_1 : D_1 \mapsto [0,1]$. *A **fuzzy goal** for Player II is similarly a fuzzy set* $\tilde{G}_2$ *represented by the MF* $\mu_2 : D_2 \mapsto [0,1]$.

An equilibrium solution is defined w.r.t. the degree of attainment of the fuzzy goals.

Definition 4 *A pair* $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in S^m \times S^n$ *is an **equilibrium solution** if, for other strategies, we have*

$$\begin{aligned} \mu_1(\boldsymbol{x}'^* \boldsymbol{A} \, \boldsymbol{y}^*) &\geq \mu_1(\boldsymbol{x}' \boldsymbol{A} \boldsymbol{y}^*), \textit{ for all } \boldsymbol{x} \in S^m \\ \mu_2(\boldsymbol{x}'^* \boldsymbol{B} \boldsymbol{y}^*) &\geq \mu_2(\boldsymbol{x}'^* \boldsymbol{B} \boldsymbol{y}), \textit{ for all } \boldsymbol{y} \in S^n. \end{aligned}$$

[1] In a min-max problem, a function to be maximized w.r.t. the maximizer variables is minimized w.r.t. the minimizer variables. Shimizu and Aiyashi [19] present necessary conditions and techniques based on the relaxation procedure for a min-max solution.

According to the Nishizaki-Sakawa model, the expression of the linear MF of the fuzzy goal $\tilde{G}_1$ for Player I may be

$$\mu_1(\mathbf{x}'\mathbf{A}\mathbf{y}) = \begin{cases} 1, \ \mathbf{x}'\mathbf{A}\mathbf{y} \geq \bar{a} \\ \frac{\mathbf{x}'\mathbf{A}\mathbf{y} - \underline{a}}{\bar{a} - \underline{a}}, \ \mathbf{x}'\mathbf{A}\mathbf{y} \in (\underline{a}, \bar{a}) \\ 0, \ \mathbf{x}'\mathbf{A}\mathbf{y} \leq \underline{a}, \end{cases}$$

where $\underline{a}$ denotes the worst degree of satisfaction of Player I, whereas $\bar{a}$ denotes his best degree of satisfaction. These values are defined as

$$\underline{a} = \min_{\mathbf{x} \in X} \min_{\mathbf{y} \in Y} \mathbf{x}'\mathbf{A}\mathbf{y} = \min_i \min_j a_{ij}, \quad (1)$$

$$\bar{a} = \max_{\mathbf{x} \in X} \max_{\mathbf{y} \in Y} \mathbf{x}'\mathbf{A}\mathbf{y} = \max_i \max_j a_{ij}. \quad (2)$$

The expression of the linear MF of the fuzzy goal $\tilde{G}_2$ for Player II will be, as well

$$\mu_2(\mathbf{x}'\mathbf{B}\mathbf{y}) = \begin{cases} 1, \ \mathbf{x}'\mathbf{B}\mathbf{y} \geq \bar{b} \\ \frac{\mathbf{x}'\mathbf{B}\mathbf{y} - \underline{b}}{\bar{b} - \underline{b}}, \ \mathbf{x}'\mathbf{B}\mathbf{y} \in (\underline{b}, \bar{b}) \\ 0, \ \mathbf{x}'\mathbf{B}\mathbf{y} \leq \underline{b}, \end{cases}$$

where $\underline{b}$ and $\bar{b}$ also denote the worst and the best degree of satisfaction of Player II, respectively. These values are deduced from Eqs.(1-2)similarly, using **B**.

Theorem 5 (Equilibrium solution) *An equilibrium solution* $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ *of the fuzzy bimatrix game, is deduced from the optimal solution* $(\boldsymbol{x}^*, \boldsymbol{y}^*, p^*, q^*)$ *of the QP problem*

$$\left. \begin{array}{rr} \max_{\boldsymbol{x},\boldsymbol{y},p,q} & \boldsymbol{x}'(\hat{\boldsymbol{A}}+\hat{\boldsymbol{B}})\boldsymbol{y} - p - q \\ & \textit{subject to} \\ & \hat{\boldsymbol{B}}'\boldsymbol{x} \leq q\boldsymbol{e}_n, \\ & \hat{\boldsymbol{A}}\boldsymbol{y} \leq p\boldsymbol{e}_m, \\ & \boldsymbol{e}'_m \boldsymbol{x} = 1, \\ & \boldsymbol{e}'_n \boldsymbol{y} = 1, \\ & \boldsymbol{x} \geq 0, \, \boldsymbol{y} \geq 0, \end{array} \right]$$

where $\hat{\mathbf{A}} = \mathbf{A}/(\bar{a} - \underline{a})$ and $\hat{\mathbf{B}} = \mathbf{B}/(\bar{b} - \underline{b})$.
Proof: see Bector and Chandra [1], pages 179-180.∎

1.4 Numerical example

In the following game [2], Player I has three pure strategies and Player II four strategies. The payoffs of Players I and II are respectively

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 7 & 2 \\ 3 & 6 & 1 & 8 \\ 2 & 5 & 3 & 9 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 5 & 1 & 2 & 4 \\ 3 & 4 & 8 & 3 \\ 1 & 8 & 1 & 2 \end{pmatrix}$$

[2] This numerical example is adapted from Nishizaki and Sakawa [17], pages 93–95.

The values of the worst and the best degree of satisfaction are given by $\underline{a} = \underline{b} = 1,\ \bar{a} = 9,\ \bar{b} = 8$. We have the QP problem

$$
\left.
\begin{array}{r}
\max_{\mathbf{x},\mathbf{y},p,q}\ \frac{1}{8}\mathbf{x}'\begin{pmatrix} 1 & 4 & 7 & 2 \\ 3 & 6 & 1 & 8 \\ 2 & 5 & 3 & 9 \end{pmatrix}\mathbf{y} \\
+\frac{1}{7}\mathbf{x}'\begin{pmatrix} 5 & 1 & 2 & 4 \\ 3 & 4 & 8 & 3 \\ 1 & 8 & 1 & 2 \end{pmatrix}\mathbf{y} \\
-p-q \\
\text{subject to} \\
\frac{1}{7}\begin{pmatrix} 5 & 3 & 1 \\ 1 & 4 & 8 \\ 2 & 8 & 1 \\ 4 & 3 & 2 \end{pmatrix}\mathbf{x} \leq q\,\mathbf{e}_4, \\
\frac{1}{8}\begin{pmatrix} 1 & 4 & 7 & 2 \\ 3 & 6 & 1 & 8 \\ 2 & 5 & 3 & 9 \end{pmatrix}\mathbf{y} \leq p\,\mathbf{e}_3, \\
\mathbf{e}_3'\mathbf{x} = 1, \\
\mathbf{e}_4'\mathbf{y} = 1, \\
\mathbf{x} \geq 0,\ \mathbf{y} \geq 0.
\end{array}
\right]
$$

The optimum solutions of the QP problem are [3] $\mathbf{x}^* = (.625, .375, 0.),\ \mathbf{y}^* = (.75, 0., .20, 0.),\ p^* = .3125,\ q^* = .6071$.

2 Single objective fuzzy bimatrix game

A single objective bimatrix game is now played in a fuzzy environment where both objectives and payoffs are uncertain. The fuzzy goals and payoffs are characterized by linear MFs. The equilibrium solutions are evaluated w.r.t. the degree of attainment of the fuzzy goals, as in the Nishizaki-Sakawa model. The optimal solutions are those of nonlinear programming problems [4].

2.1 Membership functions

The linear MF of the Player I is defined as

$$
\mu_{\tilde{G}_1}(p) = \begin{cases} 1, & p > \bar{a} \\ (p-\underline{a})/(\bar{a}-\underline{a}), & p \in (\underline{a}, \bar{a}) \\ 0, & p \leq \underline{a} \end{cases}
$$

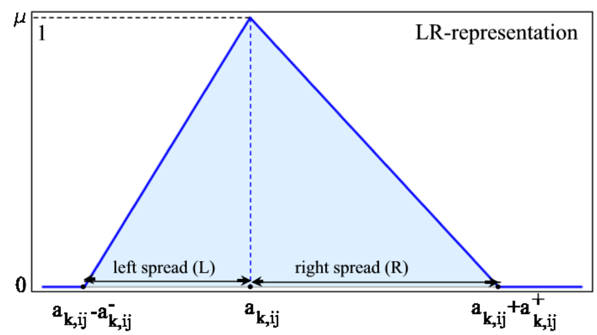

Figure 1: Fuzzy LR-type payoffs

where $p \in D_1$ denotes the Player I's fuzzy goal, $\underline{a}$ is the worst degree of satisfaction of Player I, whereas $\bar{a}$ denotes the best degree of satisfaction of Player I [5]. Similarly, the linear MF of the Player II's fuzzy goal is

$$
\mu_{\tilde{G}_2}(p) = \begin{cases} 1, & p > \bar{b} \\ (p-\underline{b})/(\bar{b}-\underline{b}), & p \in (\underline{b}, \bar{b}) \\ 0, & p \leq \underline{b} \end{cases}
$$

where $p \in D_2$ denotes the Player II's fuzzy goal, $\underline{b}$ is the worst degree of satisfaction of Player II, whereas $\bar{b}$ denotes the best degree of satisfaction of Player II. Let the fuzzy payoffs have an LR-representation, where the shape function is

$$
L(p) = R(p) = \max\{0, 1 - |p|\}.
$$

The fuzzy entries $\tilde{a}_{ij}$ of the matrix $\widetilde{\mathbf{A}}$ are

$$
\tilde{a}_{ij} = (a_{ij}, \delta^-_{a_{ij}}, \delta^+_{a_{ij}})_{LR},
$$

where a_{ij} denotes the mean value, $\delta^-_{a_{ij}}$ and $\delta^+_{a_{ij}}$ the left and right spreads, respectively (see Figure 1). The fuzzy entries $\tilde{a}_{ij}$ are characterized by the MF

$$
\mu_{\tilde{a}_{ij}}(p) = \begin{cases} 0, & p < a_{ij} - \delta^-_{a_{ij}} \\ \frac{p - a_{ij} + \delta^-_{a_{ij}}}{\delta^-_{a_{ij}}}, & p \in [\, a_{ij} - \delta^-_{a_{ij}}, a_{ij}) \\ \frac{a_{ij} + \delta^+_{a_{ij}} - p}{\delta^+_{a_{ij}}}, & p \in [a_{ij}, a_{ij} + \delta^+_{a_{ij}}) \\ 0, & p > a_{ij} + \delta^+_{a_{ij}}, \end{cases}
$$

Definition 6 *Based on the principle of decision by Bellman and Zadeh [2] [6], the* ***fuzzy decision*** *is the*

[3] The QP problem is solved by using the primitive 'NMaximize' of the *MATHEMATICA* package for searching a global maximum.

[4] This presentation is inspired by Nishizaki and Sakawa [17], pages 103-108, with adapted notations.

[5] As mentioned, these values are calculated by $\underline{a} = \min_i \min_j a_{ij}$ and $\bar{a} = \max_i \max_j a_{ij}$.

[6] See, appendix C.

intersection of the the fuzzy goals and fuzzy expected payoffs [7], *such as for Player I, we have*

$$\mu_{a(\boldsymbol{x},\boldsymbol{y})} = \min\left\{\mu_{\boldsymbol{x}'\widetilde{\boldsymbol{A}}\boldsymbol{y}}(p), \mu_{\tilde{G}_1}(p)\right\}$$

and for Player II,

$$\mu_{b(\boldsymbol{x},\boldsymbol{y})} = \min\left\{\mu_{\boldsymbol{x}'\widetilde{\boldsymbol{B}}\boldsymbol{y}}(p), \mu_{\tilde{G}_2}(p)\right\}.$$

Definition 7 *A* ***degree of attainment of the fuzzy goal*** *is defined as the maximum of the MF $\mu_{a(\boldsymbol{x},\boldsymbol{y})}$. We have*

$$d_1(\boldsymbol{x},\boldsymbol{y}) = \max_p\left(\min\left\{\mu_{\boldsymbol{x}'\widetilde{\boldsymbol{A}}\boldsymbol{y}}(p), \mu_{\tilde{G}_1}(p)\right\}\right).$$

The degree of attainment of the fuzzy goal for Player II $d_2(\boldsymbol{x},\boldsymbol{y})$ is similarly defined as

$$d_2(\boldsymbol{x},\boldsymbol{y}) = \max_p\left(\min\left\{\mu_{\boldsymbol{x}'\widetilde{\boldsymbol{B}}\boldsymbol{y}}(p), \mu_{\tilde{G}_2}(p)\right\}\right).$$

Definition 8 *For any pair of strategies $(\boldsymbol{x},\boldsymbol{y})$, the* ***degree of attainment of the fuzzy goal*** *for Player I and Player II are respectively*

$$d_1(\boldsymbol{x},\boldsymbol{y}) = \frac{\boldsymbol{x}'(\boldsymbol{A}+\Delta_{\boldsymbol{A}})\boldsymbol{y} - \underline{a}}{\bar{a} - \underline{a} + \boldsymbol{x}'\Delta_{\boldsymbol{A}}\boldsymbol{y}}$$

and

$$d_2(\boldsymbol{x},\boldsymbol{y}) = \frac{\boldsymbol{x}'(\boldsymbol{B}+\Delta_{\boldsymbol{B}})\boldsymbol{y} - \underline{b}}{\bar{b} - \underline{b} + \boldsymbol{x}'\Delta_{\boldsymbol{B}}\boldsymbol{y}},$$

where $\Delta_{\boldsymbol{A}}$ (resp. $\Delta_{\boldsymbol{B}}$) denotes the right spread matrix of the fuzzy matrix $\widetilde{\boldsymbol{A}}_{LR}$ (resp. $\widetilde{\boldsymbol{B}}_{LR}$).

2.2 Nash equilibrium solution

According to the Nishizaki and Sakawa's model, each player is supposed to maximize the degree of attainment of his goal. An equilibrium solution is then defined w.r.t. the degree of attainment of the fuzzy goals by the two players.

Definition 9 *Let $G = (S^m, S^n, \widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}})$ be a fuzzy bimatrix game, the* ***Nash equilibrium solution*** *w.r.t. the degree of attainment of the fuzzy goal is a pair of strategies $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ if, for all other strategies, we have*

$$\begin{aligned} d_1(\boldsymbol{x}^*,\boldsymbol{y}^*) &\geq d_1(\boldsymbol{x},\boldsymbol{y}^*) \text{ for all } \boldsymbol{x} \in S^m, \\ d_2(\boldsymbol{x}^*,\boldsymbol{y}^*) &\geq d_2(\boldsymbol{x}^*,\boldsymbol{y}) \text{ for all } \boldsymbol{y} \in S^n. \end{aligned}$$

The Player I's programming problem is

$$\left.\begin{array}{r} \max_{\mathbf{x}} \quad d_1(\mathbf{x},\mathbf{y}^*) = \frac{\mathbf{x}'(\mathbf{A}+\Delta_{\mathbf{A}})\mathbf{y}^* - \underline{a}}{\bar{a}-\underline{a}+\mathbf{x}'\Delta_{\mathbf{A}}\mathbf{y}^*} \\ \text{subject to} \\ \mathbf{e}'_m\mathbf{x} = 1, \\ \mathbf{x} \geq 0. \end{array}\right]$$

The Player II's programming problem is

$$\left.\begin{array}{r} \max_{\mathbf{y}} \quad d_2(\mathbf{x}^*,\mathbf{y}) = \frac{\mathbf{x}'^*(\mathbf{B}+\Delta_{\mathbf{B}})\mathbf{y} - \underline{b}}{\bar{b}-\underline{b}+\mathbf{x}'^*\Delta_{\mathbf{B}}\mathbf{y}} \\ \text{subject to} \\ \mathbf{e}'_n\mathbf{y} = 1, \\ \mathbf{y} \geq 0. \end{array}\right]$$

Applying the Kuhn-Tucker necessary and sufficient conditions, we have the equivalence Theorem 10:

Theorem 10 (Equivalence Theorem) *Let $G = (S^m, S^n, \widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}})$ be a fuzzy bimatrix game, a necessary and sufficient condition that $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ be an equilibrium point, is the solution of the non linear programming problem*

$$\left.\begin{array}{r} \max_{\boldsymbol{x},\boldsymbol{y},\psi,\xi} \quad \bar{a}\,\boldsymbol{x}'(\boldsymbol{A}+\Delta_{\boldsymbol{A}})\boldsymbol{y} + \bar{b}\,\boldsymbol{x}'(\boldsymbol{B}+\Delta_{\boldsymbol{B}})\boldsymbol{y} \\ -\underline{a}\,\boldsymbol{x}'\boldsymbol{A}\boldsymbol{y} - \psi(\bar{a}-\underline{a}+\boldsymbol{x}'\Delta_{\boldsymbol{A}}\boldsymbol{y})^2 \\ -\underline{b}\,\boldsymbol{x}'\boldsymbol{B}\boldsymbol{y} - \xi(\bar{b}-\underline{b}+\boldsymbol{x}'\Delta_{\boldsymbol{B}}\boldsymbol{y})^2 \\ \textit{subject to} \\ (\bar{a}-\underline{a}+\boldsymbol{x}'\Delta_{\boldsymbol{A}}\boldsymbol{y})\boldsymbol{A}_1\boldsymbol{y} + (\bar{a}-\boldsymbol{x}'\boldsymbol{A}\boldsymbol{y})(\Delta_{\boldsymbol{A}})_1\boldsymbol{y} \\ -\psi(\bar{a}-\underline{a}+\boldsymbol{x}'\Delta_{\boldsymbol{A}}\boldsymbol{y})^2 \leq 0, \\ (\bar{a}-\underline{a}+\boldsymbol{x}'\Delta_{\boldsymbol{A}}\boldsymbol{y})\boldsymbol{A}_2\boldsymbol{y} + (\bar{a}-\boldsymbol{x}'\boldsymbol{A}\boldsymbol{y})(\Delta_{\boldsymbol{A}})_2\boldsymbol{y} \\ -\psi(\bar{a}-\underline{a}+\boldsymbol{x}'\Delta_{\boldsymbol{A}}\boldsymbol{y})^2 \leq 0, \\ (\bar{b}-\underline{b}+\boldsymbol{x}'\Delta_{\boldsymbol{B}}\boldsymbol{y})\boldsymbol{B}_1\boldsymbol{y} + (\bar{b}-\boldsymbol{x}'\boldsymbol{B}\boldsymbol{y})(\Delta_{\boldsymbol{B}})_1\boldsymbol{y} \\ -\xi(\bar{b}-\underline{b}+\boldsymbol{x}'\Delta_{\boldsymbol{B}}\boldsymbol{y})^2 \leq 0, \\ (\bar{b}-\underline{b}+\boldsymbol{x}'\Delta_{\boldsymbol{B}}\boldsymbol{y})\boldsymbol{B}_2\boldsymbol{y} + (\bar{b}-\boldsymbol{x}'\boldsymbol{B}\boldsymbol{y})(\Delta_{\boldsymbol{B}})_2\boldsymbol{y} \\ -\xi(\bar{b}-\underline{b}+\boldsymbol{x}'\Delta_{\boldsymbol{B}}\boldsymbol{y})^2 \leq 0, \\ \boldsymbol{e}'_m\boldsymbol{x} = 1, \\ \boldsymbol{e}'_n\boldsymbol{y} = 1, \\ \boldsymbol{x} \geq 0,\ \boldsymbol{y} \geq 0, \end{array}\right]$$

where ψ, ξ are scalars, $\boldsymbol{A}_i$ and $\boldsymbol{B}_i,\ i = 1,2$ are the ith row of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively.

Proof: see Nishizaki and Sakawa [17], pages 105-107 [8]. ■

2.3 α-Nash equilibrium solution

Given a bimatrix game $G = (S^m, S^n, \widetilde{\mathbf{A}}, \widetilde{\mathbf{B}})$ with fuzzy payoffs. Suppose that the payoffs are triangular fuzzy numbers (TFNs) of the form $\tilde{a} = (l, m, u)$,

[7] Nishizaki and Sakawa [16] also consider a convex combination.

[8] The notations of Nishizaki and Sakawa [17] , pages 105–108 are those of this article except for the right spread matrices $\grave{A}$ and $\grave{B}$ which are denoted by Δ_A and Δ_B.

where the real numbers l, m and u denote the lower, the middle and the upper value, respectively. The fuzzy payoffs of Player I are represented by $\widetilde{\mathbf{A}} = (\tilde{a}_{ij})_{m\times n}$. The entry $\tilde{a}_{ij}$ denotes the (fuzzy) payoff that Player I receives when the Players I and II choose the pure strategy i and j respectively.

Definition 11 *The α-**cut** of a fuzzy number $\tilde{a}$ is defined by $a_\alpha = \{x \in X \,|\mu_{\tilde{a}}(x) \geq \alpha\}$. It can be represented by the closed interval*

$$[\underline{a}_\alpha, \bar{a}_\alpha] = \{\lambda(\bar{a}_\alpha - \underline{a}_\alpha) + \underline{a}_\alpha,\ \lambda \in [0,1]\},$$

where $\underline{a}_\alpha$ and $\bar{a}_\alpha$ denote the real lower and the upper bounds of the elements, respectively.

According to the method for solving classical games under uncertainty, Larbani [11] introduces Nature as a third Player : Nature chooses the payoffs of Players I and II and the two players express their beliefs about the behavior of Nature. The α-cuts of the payoffs of Player I are

$$\begin{aligned} \tilde{\mathbf{A}}_{\alpha_1} &= [\underline{\mathbf{A}}_{\alpha_1}, \bar{\mathbf{A}}_{\alpha_1}], \\ &= \Big\{\Lambda(\bar{\mathbf{A}}_{\alpha_1} - \underline{\mathbf{A}}_{\alpha_1}) + \underline{\mathbf{A}}_{\alpha_1}\Big\}, \end{aligned}$$

where $\Lambda = (\lambda_{ij})_{m\times n} \in [0,1]$. Similarly, α-cuts of the payoffs of Player II are

$$\begin{aligned} \tilde{\mathbf{B}}_{\alpha_2} &= [\underline{\mathbf{B}}_{\alpha_2}, \bar{\mathbf{B}}_{\alpha_2}], \\ &= \Big\{\Pi(\bar{\mathbf{B}}_{\alpha_2} - \underline{\mathbf{B}}_{\alpha_2}) + \underline{\mathbf{B}}_{\alpha_2}\Big\}, \end{aligned}$$

where $\Pi = (\pi_{ij})_{m\times n} \in [0,1]$, $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$. Nature will be favorable to Player I (resp. to Player II) if $\lambda_{ij} \in [\frac{1}{2}, 1]$ (resp. $\pi_{ij} \in [\frac{1}{2}, 1]$) and Nature will be unfavorable to those players, otherwise. For the extreme values $\lambda_{ij} = 0$ (resp. $\pi_{ij} = 0$), Player I (resp. Player II) is rather strong pessimistic. For $\lambda_{ij} = 1$ (resp. $\pi_{ij} = 1$) Player I (resp. Player II) is rather strong optimistic. If $\lambda_{ij} = \pi_{ij} = \frac{1}{2}$, Nature has a balanced behavior towards the players (Larbani [11]). The solution can be found by solving the QP problem

$$\left.\begin{array}{rr} \max_{\mathbf{x},\mathbf{y},p,q} & \mathbf{x}'\big(\mathbf{A}(\lambda^0) + \mathbf{B}(\pi^0)\big)\mathbf{y} - p - q \\ & \text{subject to} \\ & \mathbf{B}_i(\pi^0)\mathbf{x} \leq q\mathbf{e}_n,\ i = 1,2 \\ & \mathbf{A}_j(\lambda^0)\mathbf{y} \leq p\mathbf{e}_m,\ j = 1,2 \\ & \mathbf{e}'_m\mathbf{x} = 1, \\ & \mathbf{e}'_n\mathbf{y} = 1, \\ & \mathbf{x} \geq 0,\ \mathbf{y} \geq 0, \end{array}\right]$$

Proposition 12 **(α-Nash equilibrium)** *Let T_{ij} and U_{ij} be closed subsets for λ_{ij} and π_{ij} respectively in $[0,1]$. An α-Nash equilibrium $(\boldsymbol{x}^*, \boldsymbol{y}^*, \lambda^0, \pi^0)$ of the game $G = \big(S^m, S^n, \boldsymbol{A}(\lambda^0), \boldsymbol{B}(\pi^0)\big)$ is such that $\lambda^0 = \min\, T_{ij}$ and $\pi^0 = \min\, U_{ij}$.*

Proof: Larbani [11], page 661. ■

2.4 Numerical example

2.4.1 Nash equilibrium

In the following two-players example [9], Players I and II have two pure strategies. The goals of the two players are fuzzy. The payoffs are TFNs. The LR-representations of the payoffs are the tensors $\widetilde{\mathbf{A}} \in \mathbb{R}^{2\times 2\times 3}$ and $\widetilde{\mathbf{B}} \in \mathbb{R}^{2\times 2\times 3}$ for Players I and II respectively, are

$$\widetilde{\mathbf{A}}_{LR} = \begin{pmatrix} (180,5,10) & (156,6,2) \\ (90,10,10) & (180,5,10) \end{pmatrix}$$

and

$$\widetilde{\mathbf{B}}_{LR} = \begin{pmatrix} (200,10,15) & (132,4,6) \\ (120,5,10) & (156,6,6) \end{pmatrix}.$$

The right spread matrices are

$$\Delta_{\mathbf{A}} = \begin{pmatrix} 10 & 2 \\ 10 & 10 \end{pmatrix} \text{ and } \Delta_{\mathbf{B}} = \begin{pmatrix} 15 & 6 \\ 10 & 6 \end{pmatrix}.$$

The optimal solutions of Player I are $x_1^* = .2366$ and $x_2^* = .7634$ w.r.t. a degree of attainment of the goal [10] of 75.3 per cent. The optimal solutions of Player II are $y_1^* = .2963$ and $y_2^* = .7037$ w.r.t. a degree of attainment of the goal of 39.4 per cent.

2.4.2 α-Nash equilibrium

Larbani [11] introduces the beliefs of the players about the possible values of the payoffs. The α-cuts of the payoffs of Player I are defined by $\mathbf{A}_{\alpha_1} = [\underline{\mathbf{A}}_{\alpha_1}, \bar{\mathbf{A}}_{\alpha_1}]$, where the lower and upper bound matrices $\underline{\mathbf{A}}_{\alpha_1}$ and $\bar{\mathbf{A}}_{\alpha_1}$ denote the lower and upper bound matrices.

$$\underline{\mathbf{A}}_{\alpha_1} = \begin{pmatrix} 175 + 5\alpha_1 & 150 + 6\alpha_1 \\ 80 + 10\alpha_1 & 175 + 5\alpha_1 \end{pmatrix},$$

$$\bar{\mathbf{A}}_{\alpha_1} = \begin{pmatrix} 190 - 10\alpha_1 & 158 - 2\alpha_1 \\ 100 - 10\alpha_1 & 190 - 10\alpha_1 \end{pmatrix}.$$

The α-cuts of the payoffs of Player II are similarly defined by $\mathbf{B}_{\alpha_2} = [\underline{\mathbf{B}}_{\alpha_2}, \bar{\mathbf{B}}_{\alpha_2}]$, where $\underline{\mathbf{B}}_{\alpha_2}$ and $\bar{\mathbf{B}}_{\alpha_2}$ are the lower and upper bound matrices.

$$\underline{\mathbf{B}}_{\alpha_2} = \begin{pmatrix} 190 + 10\alpha_2 & 128 + 4\alpha_2 \\ 115 + 5\alpha_2 & 150 + 6\alpha_2 \end{pmatrix},$$

[9] This numerical application is an extension of the Campos's example [4].

[10] We have $d_1^* = \frac{\mathbf{x}^*(\mathbf{A}+\Delta_{\mathbf{A}})\mathbf{y}^* - a}{\bar{a} - \underline{a} + \mathbf{x}^*\Delta_{\mathbf{A}}\mathbf{y}^*} = .7532$.

$$\bar{\mathbf{B}}_{\alpha_2} = \begin{pmatrix} 215 - 15\alpha_2 & 138 - 6\alpha_2 \\ 130 - 10\alpha_2 & 162 - 6\alpha_2 \end{pmatrix}.$$

If the players choose α-cut levels such as $\alpha_1 = \alpha_2 = \frac{1}{2}$, the α-cut matrices of Player I and Player II are respectively

$$\mathbf{A}_{\frac{1}{2}} = \begin{pmatrix} [175.5, 189] & [150.6, 157.8] \\ [81, 99] & [175.5, 189] \end{pmatrix}$$

and

$$\mathbf{B}_{\frac{1}{2}} = \begin{pmatrix} [191, 213.5] & [128.4, 137.4] \\ [115.5, 129] & [156.6, 161.4] \end{pmatrix}$$

Suppose, as in Larbani [11], that the players may have two types of beliefs. In the **first case**, the players believe that Nature plays against them. In the **second case**, Player I believes that Nature is favorable to him, only for the pairs of strategies (1,1) and (2,2), and against him for the other pairs of strategies. Player II still believes that Nature is against him for all pairs of strategies. The **third case** shows the dependence between the profits of the two players and the strategies chosen by Nature.

case 1: For Player I, we have $T_{ij} = U_{ij} = [0, \frac{1}{3}]$ and $\lambda^0 = \pi^0 = 0$. The payoff matrices of Players I and II are respectively

$$\mathbf{A}(\lambda^0) = \begin{pmatrix} \frac{355}{2} & 153 \\ 85 & \frac{355}{2} \end{pmatrix}$$

and

$$\mathbf{B}(\pi^0) = \begin{pmatrix} 195 & 130 \\ \frac{235}{2} & 153 \end{pmatrix}.$$

The game has three Nash equilibria [11].The game has two perfect equilibria and one mixed equilibrium. The first perfect Nash equilibrium is

$$(x_1^*, x_2^*) = (0, 1), (y_1^*, y_2^*) = (0, 1)$$

with an expected payoff of 177.5 for Player I and an expected payoff of 153 for Player II. The second perfect Nash equilibrium is

$$(x_1^*, x_2^*) = (1, 0), (y_1^*, y_2^*) = (1, 0)$$

with an expected payoff of 177.5 for Player I and an expected payoff of 195 for Player II. The third mixed Nash equilibrium is

$$(x_1^*, x_2^*) = (.3532, .6468), (y_1^*, y_2^*) = (.2094, .7906)$$

with an expected payoff of 158.1 for Player I and an expected payoff of 144.9 for Player II.

[11]The package Game Theory of *MATHEMATICA* is used to calculate all the perfect and mixed Nash equilibrium (Canty [5]).

case 2: For Player I, we have

$$T_{ij} = \begin{cases} [\frac{2}{3}, 1], \ i = j, \ i, j \in \{1, 2\} \\ 0, \ \text{otherwise.} \end{cases}$$

then, we have

$$\lambda^0 = \begin{cases} \frac{2}{3}, \ i = j \in \{1, 2\} \\ 0, \ \text{otherwise.} \end{cases}$$

The Player II's beliefs are similar to those of the case 1. The payoff matrices of Players I and II are respectively

$$\mathbf{A}(\lambda^0) = \begin{pmatrix} \frac{365}{2} & \frac{467}{3} \\ \frac{275}{3} & \frac{365}{2} \end{pmatrix}$$

and

$$\mathbf{B}(\pi^0) = \begin{pmatrix} 195 & 130 \\ \frac{235}{2} & 153 \end{pmatrix}.$$

The game has three Nash equilibria. Two of them are perfect equilibria and one is a mixed equilibrium. The first perfect Nash equilibrium is

$$(x_1^*, x_2^*) = (0, 1), (y_1^*, y_2^*) = (0, 1)$$

with an expected payoff of 182.5 for Player I and an expected payoff of 153 for Player II. The second perfect Nash equilibrium is

$$(x_1^*, x_2^*) = (1, 0), (y_1^*, y_2^*) = (1, 0)$$

with an expected payoff of 182.5 for Player I and an expected payoff of 195 for Player II. The third mixed Nash equilibrium is

$$(x_1^*, x_2^*) = (.3532, .6468), (y_1^*, y_2^*) = (.2280, .7720)$$

with an expected payoff of 161.8 for Player I and an expected payoff of 144.9 for Player II.

case 3: The profits of Players I and II depend on the strategies that Nature will choose. Nature is favorable to the players in the range $[\frac{1}{2}, 1]$ for λ and μ. The resulting profits in varying λ and μ in the interval $[0, 1]$ are illustrated by the density plots Figure 2. The profits are then increasing when Nature is more favorable (with higher values of λ and μ).

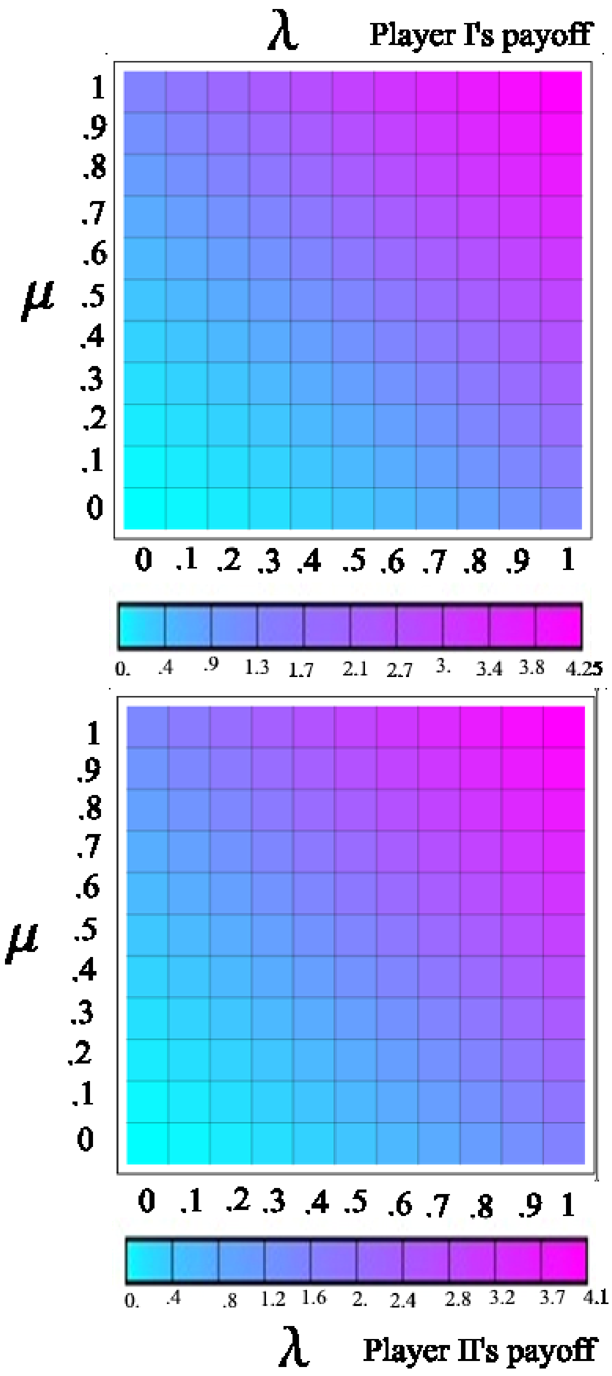

Figure 2: Players' profits and Nature's strategies

3 Multiobjective fuzzy bimatrix game

A multiple objectives bimatrix game is presented [12] in a fuzzy environment where both the objectives and the payoffs are uncertain. The list of the r payoff matrices for Player I is represented by $\widetilde{\mathbf{A}}^k = (\tilde{a}^k_{ij})_{m\times n},\ k \in \mathbb{N}_r$, with fuzzy entries. The list of the s payoff matrices for Player II is represented by $\widetilde{\mathbf{B}}^l = (\tilde{b}^l_{ij})_{m\times n},\ l \in \mathbb{N}_s$, with fuzzy entries.

3.1 Fuzzy expected payoff

For triangular fuzzy payoffs, we have the following LR-representations of entries $\tilde{a}^k_{ij} = \left(a^k_{ij}, \delta^{k-}_{a_{ij}}, \delta^{k+}_{a_{ij}}\right)_{LR}$ and $\tilde{b}^l_{ij} = \left(b^l_{ij}, \delta^{l-}_{b_{ij}}, \delta^{l+}_{b_{ij}}\right)_{LR}$.

Definition 13 *For any pair of mixed strategies $(\boldsymbol{x},\boldsymbol{y})$, the kth* ***fuzzy expected payoff*** *for Player I is defined by*

$$\boldsymbol{x}'\widetilde{\boldsymbol{A}}^k\boldsymbol{y} = \left(\boldsymbol{x}'\boldsymbol{A}^k\boldsymbol{y}, \boldsymbol{x}'\Delta^{k-}_{\boldsymbol{A}}\boldsymbol{y}, \boldsymbol{x}'\Delta^{k+}_{\boldsymbol{A}}\boldsymbol{y}\right)_{LR}$$

and is characterized by the MF

$$\mu_{\boldsymbol{x}'\widetilde{\boldsymbol{A}}^k\boldsymbol{y}} : D^k_1 \mapsto [0,1],$$

where $D^k_1 \subseteq \mathbb{R}$ denote the domain of the kth payoff for Player I. The lth fuzzy expected payoff of Player II is similarly defined by

$$\boldsymbol{x}'\widetilde{\boldsymbol{B}}^l\boldsymbol{y} = \left(\boldsymbol{x}'\boldsymbol{B}^l\boldsymbol{y}, \boldsymbol{x}'\Delta^{l-}_{\boldsymbol{B}}\boldsymbol{y}, \boldsymbol{x}'\Delta^{l+}_{\boldsymbol{B}}\boldsymbol{y}\right)_{LR}$$

and is characterized by the MF

$$\mu_{\boldsymbol{x}'\widetilde{\boldsymbol{B}}^l\boldsymbol{y}} : D^l_2 \mapsto [0,1],$$

where $D^l_2 \subseteq \mathbb{R}$ denotes the domain of the lth payoff for Player II.

3.2 Fuzzy goal attainment

Definition 14 *Let the* ***fuzzy goals*** *of Players I and II be denoted by $\boldsymbol{p}_1 = (p^1_1, \ldots, p^r_1) \in D_1 \subseteq \mathbb{R}^r$ and $\boldsymbol{p}_2 = (p^1_2, \ldots, p^s_2) \in D_2 \subseteq \mathbb{R}^s$. The Player I's kth fuzzy goal G^k_1 is a fuzzy set characterized by the MF*

$$\mu^k_1 : D^k_1 \mapsto [0,1],\ k \in \mathbb{N}_r.$$

Similarly, the Player II's lth fuzzy goal G^l_2 is a fuzzy set characterized by the MF

$$\mu^l_2 : D^l_2 \mapsto [0,1],\ l \in \mathbb{N}_s.$$

[12] This presentation is inspired from Nishizaki and Sakawa [17], pages 108-114.

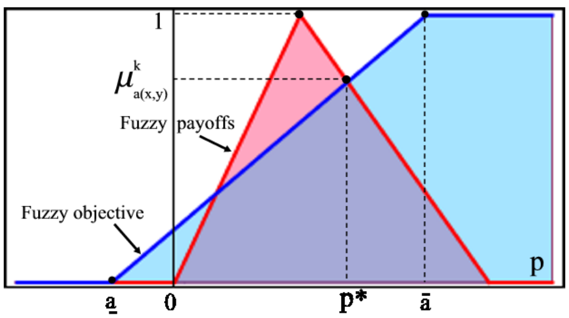

Figure 3: Degree of attainment of a fuzzy goal

Definition 15 *For any pair of strategies $(\boldsymbol{x}, \boldsymbol{y})$, an **attainment state of the fuzzy goal** is represented by the intersection of the fuzzy expected payoff $\boldsymbol{x}'\widetilde{\boldsymbol{A}}^k\boldsymbol{y}$ and the fuzzy goal $\tilde{G}_1^k$. We have*

$$\mu^k_{a(\boldsymbol{x},\boldsymbol{y})}(p) = \min\left\{\mu^k_{a(\boldsymbol{x}\widetilde{\boldsymbol{A}}^k\boldsymbol{y})}(p), \mu_{\tilde{G}_1^k}(p)\right\},$$

*where $p \in D_1^k$ is a payoff of Player I. The **degree of attainment** of the kth fuzzy goal for Player II is the maximum of the MF, such as*

$$\hat{\mu}^k_{a(\boldsymbol{x},\boldsymbol{y})}(p^*) = \max_p \mu^k_{a(\boldsymbol{x},\boldsymbol{y})}(p).$$

Similarly, the degree of attainment of the fuzzy goal for Player II is

$$\hat{\mu}^l_{b(\boldsymbol{x},\boldsymbol{y})}(p^*) = \max_p \left(\min\left\{\mu^l_{b(\boldsymbol{x}\widetilde{\boldsymbol{B}}^l\boldsymbol{y})}(p), \mu_{\tilde{G}_2^l}(p)\right\}\right).$$

The Figure 3 illustrates the concept.

3.3 Equilibrium solution

An equilibrium solution is defined w.r.t. the degree of attainment of the aggregated fuzzy goal.

Definition 16 *Let $G = (S^m, S^n, \widetilde{\boldsymbol{A}}^k, \widetilde{\boldsymbol{B}}^l, k, l)$ be a multiobjective fuzzy bimatrix game, and denote the degrees of attainment of the aggregated fuzzy goal for Players I and II by $D^1(\boldsymbol{x},\boldsymbol{y})$ and $D^2(\boldsymbol{x},\boldsymbol{y})$, respectively. The **equilibrium solution w.r.t. the degree of attainment of the aggregated fuzzy goal** is a pair of strategies $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ if, for all other strategies, we have*

$$\begin{aligned} D^1(\boldsymbol{x}^*,\boldsymbol{y}^*) &\geq D^1(\boldsymbol{x},\boldsymbol{y}^*), \text{ for all } \boldsymbol{x} \in S^m \\ D^2(\boldsymbol{x}^*,\boldsymbol{y}^*) &\geq D^2(\boldsymbol{x}^*,\boldsymbol{y}), \text{ for all } \boldsymbol{y} \in S^n. \end{aligned}$$

If the fuzzy goals are aggregated by a minimum component method, the classical decision rule by Bellman and Zadeh [2] is used [13]. This aggregation method consists in the intersection of all the fuzzy sets. The Player I's degree of attainment of the aggregated fuzzy goal is defined by

$$D^1(\mathbf{x},\mathbf{y}) = \min_{k\in\mathbb{N}_r} \frac{\mathbf{x}'(\mathbf{A}^k + \Delta^k_{\mathbf{A}})\mathbf{y} - \underline{a}^k}{\bar{a}^k - \underline{a}^k + \mathbf{x}'\Delta^k_{\mathbf{A}}\mathbf{y}}.$$

The Player I's programming problem using the kth payoffs is

$$\left.\begin{aligned} \max\nolimits_{\mathbf{x},\sigma} \quad \sigma \\ \text{subject to} \\ \tfrac{\mathbf{x}'(\mathbf{A}^k+\Delta^k_{\mathbf{A}})\mathbf{y}^* - \underline{a}^k}{\bar{a}^k - \underline{a}^k + \mathbf{x}'\Delta^k_{\mathbf{A}}\mathbf{y}^*} \geq \sigma, \\ \mathbf{e}'_m\mathbf{x} = 1, \\ \mathbf{x} \geq 0. \end{aligned}\right]$$

The Player II's programming problem for Player II using the lth payoffs is

$$\left.\begin{aligned} \max\nolimits_{\mathbf{y},\delta} \quad \delta \\ \text{subject to} \\ \tfrac{\mathbf{x}'^*(\mathbf{B}^l+\Delta^l_{\mathbf{B}})\mathbf{y} - \underline{b}^l}{\bar{b}^l - \underline{b}^l + \mathbf{x}'^*\Delta^l_{\mathbf{B}}\mathbf{y}} \geq \delta, \\ \mathbf{e}'_n\mathbf{y} = 1, \\ \mathbf{y} \geq 0. \end{aligned}\right]$$

Applying the Kuhn-Tucker necessary and sufficient conditions, we have the equivalence Theorem 17.

Theorem 17 (Equivalence Theorem) *Let $G = (S^m, S^n, \widetilde{\boldsymbol{A}}^k, \widetilde{\boldsymbol{B}}^l)$ be a multiobjective fuzzy bimatrix game, a necessary and sufficient condition that $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ be an equilibrium point is the solution of*

[13] One another method for aggregating multiple fuzzy goals is weighting the coefficients.

the nonlinear programming problem

$$\max_{\boldsymbol{x},\boldsymbol{y},\psi,\xi,\sigma,\delta,\Lambda,\Theta} \left\{ \sum_{k=1}^{r} \lambda_k \left[\frac{\underline{a}^k(2\boldsymbol{x}'\Delta_A^k\boldsymbol{y}+\bar{a}^k-\underline{a}^k)}{(\bar{a}^k-\underline{a}^k+\boldsymbol{x}'\Delta_A^k\boldsymbol{y})^2} - \frac{\boldsymbol{x}'\Delta_A^k\boldsymbol{y}\times\boldsymbol{x}'(\boldsymbol{A}^k+\Delta_A^k)\boldsymbol{y}}{(\bar{a}^k-\underline{a}^k+\boldsymbol{x}'\Delta_A^k\boldsymbol{y})^2} \right] + \sigma - \psi + \sum_{l=1}^{s} \theta_l \left[\frac{\underline{b}^l(2\boldsymbol{x}'\Delta_B^l\boldsymbol{y}+\bar{b}^l-\underline{b}^l)}{(\bar{b}^l-\underline{b}^l+\boldsymbol{x}'\Delta_B^l\boldsymbol{y})^2} - \frac{\boldsymbol{x}'\Delta_B^l\boldsymbol{y}\times\boldsymbol{x}'(\boldsymbol{B}^k+\Delta_B^l)\boldsymbol{y}}{(\bar{b}^l-\underline{b}^l+\boldsymbol{x}'\Delta_B^l\boldsymbol{y})^2} \right] + \delta - \xi \right\}$$

subject to

$$\begin{aligned}
&\sum_{k=1}^{r} \lambda_k \left[\frac{(\bar{a}^k-\underline{a}^k+\boldsymbol{x}'\Delta_A^k\boldsymbol{y})\boldsymbol{A}_1^k\boldsymbol{y}}{(\bar{a}^k-\underline{a}^k+\boldsymbol{x}'\Delta_A^k\boldsymbol{y})^2} + \frac{(\bar{a}^k-\boldsymbol{x}'\boldsymbol{A}^k\boldsymbol{y})(\Delta_A^k)_1\boldsymbol{y}}{(\bar{a}^k-\underline{a}^k+\boldsymbol{x}'\Delta_A^k\boldsymbol{y})^2} \right] - \psi \le 0,\\
&\sum_{k=1}^{r} \lambda_k \left[\frac{(\bar{a}^k-\underline{a}^k+\boldsymbol{x}'\Delta_A^k\boldsymbol{y})\boldsymbol{A}_2^k\boldsymbol{y}}{(\bar{a}^k-\underline{a}^k+\boldsymbol{x}'\Delta_A^k\boldsymbol{y})^2} + \frac{(\bar{a}^k-\boldsymbol{x}'\boldsymbol{A}^k\boldsymbol{y})(\Delta_A^k)_2\boldsymbol{y}}{(\bar{a}^k-\underline{a}^k+\boldsymbol{x}'\Delta_A^k\boldsymbol{y})^2} \right] - \psi \le 0,\\
&\sum_{l=1}^{s} \theta_l \left[\frac{(\bar{b}^l-\underline{b}^l+\boldsymbol{x}'\Delta_B^l\boldsymbol{y})(\boldsymbol{B}_1^l)'\boldsymbol{x}}{(\bar{b}^l-\underline{b}^l+\boldsymbol{x}'\Delta_B^l\boldsymbol{y})^2} + \frac{(\bar{b}^l-\boldsymbol{x}'\boldsymbol{B}^l\boldsymbol{y})(\Delta_B^l)_1'\boldsymbol{x}}{(\bar{b}^l-\underline{b}^l+\boldsymbol{x}'\Delta_B^l\boldsymbol{y})^2} \right] - \xi \le 0,\\
&\sum_{l=1}^{s} \theta_l \left[\frac{(\bar{b}^l-\underline{b}^l+\boldsymbol{x}'\Delta_B^l\boldsymbol{y})(\boldsymbol{B}_2^l)'\boldsymbol{x}}{(\bar{b}^l-\underline{b}^l+\boldsymbol{x}'\Delta_B^l\boldsymbol{y})^2} + \frac{(\bar{b}^l+\boldsymbol{x}'\boldsymbol{B}^l\boldsymbol{y})(\Delta_B^l)_2'\boldsymbol{x}}{(\bar{b}^l-\underline{b}^l+\boldsymbol{x}'\Delta_B^l\boldsymbol{y})^2} \right] - \xi \le 0,\\
&\sum_{l=1}^{s} \theta_l \left[\frac{(\bar{b}^l-\underline{b}^l+\boldsymbol{x}'\Delta_B^l\boldsymbol{y})(\boldsymbol{B}_3^l)'\boldsymbol{x}}{(\bar{b}^l-\underline{b}^l+\boldsymbol{x}'\Delta_B^l\boldsymbol{y})^2} + \frac{(\bar{b}^l+\boldsymbol{x}'\boldsymbol{B}^l\boldsymbol{y})(\Delta_B^l)_3'\boldsymbol{x}}{(\bar{b}^l-\underline{b}^l+\boldsymbol{x}'\Delta_B^l\boldsymbol{y})^2} \right] - \xi \le 0,\\
&\frac{\boldsymbol{x}'(\boldsymbol{A}^k+\Delta_A^k)\boldsymbol{y}-\underline{a}^k}{\bar{a}^k-\underline{a}^k+\boldsymbol{x}'\Delta_A^k\boldsymbol{y}} - \sigma \ge 0,\ k\in\mathbb{N}_r\\
&\frac{\boldsymbol{x}'(\boldsymbol{B}^l+\Delta_B^l)\boldsymbol{y}-\underline{b}^l}{\bar{b}^l-\underline{b}^l+\boldsymbol{x}'\Delta_B^l\boldsymbol{y}} - \delta \ge 0,\ l\in\mathbb{N}_s\\
&\boldsymbol{e}_m'\boldsymbol{x} = 1,\\
&\boldsymbol{e}_n'\boldsymbol{y} = 1,\\
&\boldsymbol{x}\ge 0,\ \boldsymbol{y}\ge 0,\ \Lambda\ge 0,\ \Theta\ge 0,
\end{aligned}$$

where ψ, ξ are scalars and $\Lambda' = (\lambda_k)_{1\times 3}$, $\Theta' = (\theta_l)_{1\times 3}$, scalar entries. The vector $\boldsymbol{A}_i^k$, $i = 1,2$ denotes the ith *row of the matrix $\boldsymbol{A}^k$ and similarly for the transposed matrix $(\boldsymbol{B}^l)_j'$, $j = 1,2,3$.*

Proof: see Nishizaki and Sakawa [17], pages 110–114. ∎

3.4 Numerical example

In the following two players example [14][15], Players I and II have respectively two and three pure strategies and three different objectives. The goals of the two players are fuzzy. The payoffs are triangular fuzzy numbers. The LR-representation of the payoffs are the tensors $\widetilde{\mathbf{A}}^k \in \mathbb{R}^{2\times3\times3}$, $k \in \mathbb{N}_3$ and $\widetilde{\mathbf{B}} \in \mathbb{R}^{2\times3\times3}$, $l \in \mathbb{N}_3$ for Players I and II respectively, are

$$\widetilde{\mathbf{A}}^1_{LR} = \begin{pmatrix} (1,.5,1) & (4,1,1) & (3,.5,1.5) \\ (2,1,1) & (4,.5,.5) & (1,1,1) \end{pmatrix},$$

$$\widetilde{\mathbf{A}}^2_{LR} = \begin{pmatrix} (4,.5,1) & (3,1,1) & (2,1,.5) \\ (1,1,1) & (5,1,.5) & (1,.5,1) \end{pmatrix},$$

$$\widetilde{\mathbf{A}}^3_{LR} = \begin{pmatrix} (2,1,1.5) & (0,0,1.5) & (1,.5,1) \\ (4,1.5,1.5) & (1,.5,.5) & (3,1,.5) \end{pmatrix}$$

and

$$\widetilde{\mathbf{B}}^1_{LR} = \begin{pmatrix} (0,0,1) & (2,1.5,1) & (2,1,1) \\ (5,.5,1) & (5,1,1) & (1,.5,.5) \end{pmatrix},$$

$$\widetilde{\mathbf{B}}^2_{LR} = \begin{pmatrix} (4,.5,1) & (2,1,1.5) & (5,1,.5) \\ (0,0,1) & (5,.5,.5) & (4,1.5,1) \end{pmatrix},$$

$$\widetilde{\mathbf{B}}^3_{LR} = \begin{pmatrix} (2,1,1.5) & (1,.5,1) & (4,1,1.5) \\ (1,.5,.5) & (0,0,1.5) & (1,1,1) \end{pmatrix}.$$

The right spread matrices for Player I are

$$\Delta^1_{\mathbf{A}} = \begin{pmatrix} 1 & 1 & 1.5 \\ 1 & .5 & 1 \end{pmatrix}, \Delta^2_{\mathbf{A}} = \begin{pmatrix} 1 & 1 & .5 \\ 1 & .5 & 1 \end{pmatrix},$$

$$\Delta^3_{\mathbf{A}} = \begin{pmatrix} 1.5 & 1.5 & 1 \\ 1.5 & .5 & .5 \end{pmatrix}.$$

The right spread matrices for Player II are

$$\Delta^1_{\mathbf{B}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & .5 \end{pmatrix}, \Delta^2_{\mathbf{B}} = \begin{pmatrix} 1 & 1.5 & .5 \\ 1 & .5 & 1 \end{pmatrix},$$

$$\Delta^3_{\mathbf{B}} = \begin{pmatrix} 1.5 & 1 & 1.5 \\ .5 & 1.5 & 1 \end{pmatrix}.$$

The optimal solutions [16] of Player I are $x_1^* = .6438$ and $x_2^* = .3562$ w.r.t. a degree of attainment of the

[14] This numerical application is an extension of the Chen's example [6].

[15] The contribution by Keller [9] introduces to the fuzzy optimization techniques, using the software *MATHEMATICA*. Simple classic economic examples are analysed.

[16] The numerical solutions have been obtained using the primitive 'Minimize' of *MATHEMATICA* for a timing of 7 minutes 46 for an Intel(R) Corel(TM)2 CPU6400@2.13 GHz.

goal [17] of 58.5 per cent. The optimal solutions of Player II are $y_1^* = .5226$, $y_2^* = .3149$ and $y_3^* = .1625$ w.r.t. a degree of attainment of the goal of 52.5 per cent.

4 Conclusion

The crisp bimatrix games have an equivalent QP problem for finding Nash equilibrium solutions. The single objective fuzzy bimatrix game have an equivalent nonlinear programming problem. The multiple objective bimatrix games have an extended nonlinear programming problem. All these problems may be solve by different ways, by using algorithms and optimization techniques (Lemke-Howson's algorithm, multipliers in Varian [21], Van de Panne's two phase method [20], symmetric Zimmermann's approach [25, 26]), genetic algorithm in Wang *et al.* [24], the relaxation procedure for min-max problems subject to separate constraints (Shimizu and Aiyoshi [19]).

A Karush- Kuhn- Tucker (KKT) Optimality Conditions [3]

Let a nonlinear programming problem be (see Boyd and Vandenberghe [3])

$$\left.\begin{array}{r} \min_{\mathbf{x}}\ f_0(\mathbf{x}) \\ \text{subject to} \\ f_i(\mathbf{x}) \leq 0,\ i \in \mathbb{N}_m \\ h_j(\mathbf{x}) = 0,\ j \in \mathbb{N}_p \end{array}\right]$$

The optimization variables are $\mathbf{x} \in \mathbb{R}^n$, the objective function is $f_0 : \mathbb{R}^n \mapsto \mathbb{R}$, the m inequality constraints are $f_i(\mathbf{x}) \leq 0,\ i \in \mathbb{N}_m$, and the p equality constraints are $h_j(\mathbf{x}) = 0,\ j \in \mathbb{N}_p$. All the functions $f_0, f_1, \ldots, f_m, h_1, h_2, \ldots, h_p$ are differentiable. The domain of the optimization problem is defined by

$$\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \bigcap \bigcap_{j=0}^{p} \mathbf{dom}\, h_j.$$

Associating the *m*-dimensional multiplier λ and the *p*-dimensional multiplier ν, we have the lagrangian

$$\mathcal{L}(\mathbf{x}, \lambda, \nu) = f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) + \sum_{j=1}^{p} \nu_j h_j(\mathbf{x}).$$

[17] We have

$$D^{1*} = \min_k \left\{ \frac{\mathbf{x}^*(\mathbf{A}^k + \Delta_{\mathbf{A}}^k)\mathbf{y}^* - \underline{a}^k}{\bar{a}^k - \underline{a}^k + \mathbf{x}^* \Delta_{\mathbf{A}}^k \mathbf{y}^*}, k \in \mathbb{N}_3 \right\} = .5840$$

.

Since $\mathbf{x}^*$ minimizes $\mathcal{L}(\mathbf{x}, \lambda^*, \nu^*)$ over $\mathbf{x}$, it follows that

$$\nabla f_0(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(\mathbf{x}^*) + \sum_{j=1}^{p} \nu_j^* \nabla h_j(\mathbf{x}) = 0.$$

Thus we have the KKT conditions at the primal point x^* and dual points (λ^*, ν^*)

$$\left.\begin{array}{r} f_i(\mathbf{x}^*) \leq 0,\ i \in \mathbb{N}_m \\ h_j(\mathbf{x}^*) = 0, j \in \mathbb{N}_p \\ \lambda_i^* \geq 0,\ i \in \mathbb{N}_m \\ \nabla f_0(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(\mathbf{x}^*) + \sum_{j=1}^{p} \nu_j^* \nabla h_j(\mathbf{x}) = 0, \end{array}\right]$$

Boyd and Vandenberghe [3](page 244) present the minimization quadratic problem

$$\left.\begin{array}{r} \min_{\mathbf{x}}\ \frac{1}{2}\mathbf{x}'\mathbf{P}\mathbf{x} + \mathbf{q}'\mathbf{x} + r \\ \text{subject to} \\ \mathbf{A}\mathbf{x} = \mathbf{b}, \end{array}\right]$$

where we have $\mathbf{P} \in \mathbb{R}^{n \times n}$ a symmetric positive semidefinite matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$. Let the multipliers be the vector ν, the KKT conditions are

$$\begin{array}{rcl} \mathbf{A}\mathbf{x}^* & = & \mathbf{b} \\ \mathbf{P}\mathbf{x}^* + \mathbf{q} - \mathbf{A}'\nu^* & = & 0. \end{array}$$

Thereafter, the optimal primal and dual variables are obtained by solving this set of $m+n$ equations in the $m+n$ variables $\mathbf{x}^*$ and ν^*.

B Proof of the Equivalence Theorem

The objectives of the Players I and II are achieved by solving the two programming problems, respectively

$$\left.\begin{array}{r} \max_{\mathbf{x}}\ \mathbf{x}'\mathbf{A}\mathbf{y}^* \\ \text{subject to} \\ \mathbf{e}_m'\mathbf{x} = 1, \\ \mathbf{x} \geq 0 \end{array}\right]$$

and

$$\left.\begin{array}{r} \max_{\mathbf{y}}\ \mathbf{x}'^*\mathbf{B}\mathbf{y} \\ \text{subject to} \\ \mathbf{e}_n'\mathbf{y} = 1, \\ \mathbf{y} \geq 0 \end{array}\right]$$

The equilibrium solution can be obtained by solving (see Chen [6], Mangasarian and Stone [13])

$$\left.\begin{array}{r} \max_{\mathbf{x},\mathbf{y}}\ \mathbf{x}'\mathbf{A}\mathbf{y}^* + \mathbf{x}'^*\mathbf{B}\mathbf{y} \\ \text{subject to} \\ \mathbf{e}_m'\mathbf{x} = 1, \\ \mathbf{e}_n'\mathbf{y} = 1, \\ \mathbf{x} \geq 0,\ \mathbf{y} \geq 0, \end{array}\right]$$

Let $p = \max_{\mathbf{x}} \mathbf{x}'\mathbf{A}\mathbf{y}^*$ and $q = \max_{\mathbf{y}} \mathbf{x}'^*\mathbf{B}\mathbf{y}$. The following inequalities are also true $p \geq \mathbf{x}'\mathbf{A}\mathbf{y}^* \geq \mathbf{x}'\mathbf{A}\mathbf{y}$, for all $\mathbf{x} \geq 0$. So, we have the simplification $p\mathbf{e}_m \geq \mathbf{A}\mathbf{y}$. We also have the inequalities $q \geq \mathbf{x}'^*\mathbf{B}\mathbf{y} \geq \mathbf{x}'\mathbf{B}\mathbf{y}$, for all $\mathbf{y} \geq 0$. So, we have the simplification $q\mathbf{e}'_n \geq \mathbf{B}'\mathbf{x}$. The QP problem is

$$\left.\begin{array}{r}\min_{\mathbf{x},\mathbf{y},p,q}\ (p - \mathbf{x}'\mathbf{A}\mathbf{y}) + (q - \mathbf{x}'\mathbf{B}\mathbf{y}) \\ \text{subject to} \\ \mathbf{B}'\mathbf{x} \leq q\mathbf{e}'_n, \\ \mathbf{A}\mathbf{y} \leq p\mathbf{e}_m, \\ \mathbf{e}'_m\mathbf{x} = 1, \\ \mathbf{e}'_n\mathbf{y} = 1, \\ \mathbf{x} \geq 0,\ \mathbf{y} \geq 0. \end{array}\right]$$

Then, the QP problem of the equivalence Theorem 2 is deduced ■

C Fuzzy decision sets

C.1 Bellman-Zadeh fuzzy decision rules

According to the Bellman-Zadeh symmetry principle, a fuzzy decision set is achieved by using an appropriate aggregation of the fuzzy sets.

Definition 18 *Let X be a set of possible actions, $\{\tilde{G}_j\ (j \in \mathbb{N}_n\}$ a set of fuzzy objectives, and $\{\tilde{C}_i\ (i \in \mathbb{N}_m\}$ the decision set is defined by*

$$\tilde{D} = \Big(\bigcap_{j=1}^{n} \tilde{G}_j\Big) \bigcap \Big(\bigcap_{i=1}^{m} \tilde{C}_i\Big),$$

with MF $\mu^1_{\tilde{D}} : X \mapsto [0,1]$ given by

$$\mu^1_{\tilde{D}}(x) = \Big(\bigwedge_{j=1}^{n} \mu_{\tilde{G}_j}(x)\Big) \bigwedge \Big(\bigwedge_{i=1}^{m} \mu_{\tilde{C}_i}(x)\Big).$$

The MFs of the aggregate fuzzy goal can be expressed as

$$\mu(x,y) = \min_{k \in \mathbb{N}_r} \Big\{ \mu_k(\mathbf{x}'\mathbf{A}^k\mathbf{y}) \Big\}.$$

Hence, we have with linear MFs

$$\mu(x,y) = \min_{k \in \mathbb{N}_r} \Big\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a^k_{ij}}{\bar{a}^k - \underline{a}^k} x_i y_j - \frac{\underline{a}^k}{\bar{a}^k - \underline{a}^k} \Big\}.$$

Considering the unequal importance of the fuzzy goals and constraints, Bellman and Zadeh also suggest another decision rule. This rule is defined the following convex combination of the fuzzy objective functions and constraints

$$\mu^2_D = \sum_{i=1}^{r} \alpha_i \mu_{G_i}(x) + \sum_{j=1}^{m} \beta_i \mu_{C_j}(x),$$

where all the nonnegative weighting coefficients α_i and β_j sum to one.

C.2 Product fuzzy decision set

The product fuzzy decision is an alternative decision set, defined by

$$\mu^3_D = \Big(\prod_{i=1}^{r} \alpha_i \mu_{G_i}(x)\Big) \times \Big(\prod_{j=1}^{m} \beta_i \mu_{C_j}(x)\Big),$$

C.3 Comparison of the fuzzy decision rules

The three types of decision sets are related by the inequalities

$$\mu^3_D(x) \leq \mu^1_D(x) \leq \mu^2_D(x).$$

The following example is taken from Sakawa [18] with one objective and one constraint. According to the fuzzy goal "x should be much larger than 10", and according to the fuzzy constraint "x should be substantially less or equal than 30". The MFs of the fuzzy objective and the fuzzy constraint are respectively defined by

$$\mu_G(x) = \begin{cases} 0, & x \leq 10 \\ 1 - \frac{1}{1+\left(\frac{x-10}{10}\right)^2}, & x > 10 \end{cases}$$

and

$$\mu_C(x) = \begin{cases} 0, & x \leq 30 \\ \frac{1}{1+\frac{x}{x-30}}, & x < 30 \end{cases}$$

The Figure 4 compares the fuzzy rules. In this examples, the maximum decisions are obtained for $(x^*_1, \mu^*_1) = (11.7549, .7549)$, $(x^*_2, \mu^*_2) = (11.3841, .8500)$ and $(x^*_3, \mu^*_3) = (11.4811, .6520)$.

D Fuzzy quadratic programming

The symmetric approach by Zimmermann [25] may be used for solving fuzzy programming problems. For this approach, membership functions are defined, by using a given aspiration level of the decision maker for the objective, and accepted tolerances for the objective and the constraint functions. An equivalent crisp QP problem is obtained with a quadratic constraint. This particular QP problem can be solved by using van de Panne 's two-phase method [20]

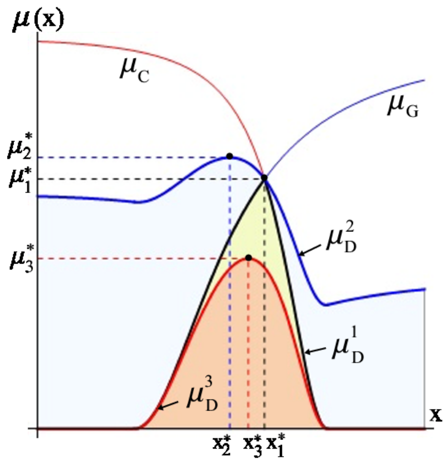

Figure 4: Fuzzy decision rules: intersection, convex and product fuzzy decision

D.1 Fuzzy QP problem

The fuzzy QP problem may be defined by a convex quadratic objective function together with a bounded feasible region such as in Bector and Chandra [1]

$$\left.\begin{array}{r} \widetilde{\min}_{\mathbf{x}}\ \mathbf{c}'\mathbf{x} + \frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x} \\ \text{subject to} \\ \mathbf{A}_i\mathbf{x} \lesssim b_i,\ i \in \mathbb{N}_m \\ \mathbf{x} \geq 0, \end{array}\right]$$

where $\mathbf{c},\ \mathbf{x} \in \mathbb{R}^n,\ \mathbf{b} \in \mathbb{R}^m,\ \mathbf{A} \in \mathbb{R}^{m\times n},\ \mathbf{Q} \in \mathbb{R}^{n\times n}$. The vector $\mathbf{A}_i$ denotes the ith row of matrix $\mathbf{A}$. The symmetric matrix $\mathbf{Q}$ is supposed to be positive semi definite.

D.2 Symmetric fuzzy QP problem

According to Zimmermann [25, 26], the symmetric version of the fuzzy QP problem is

$$\left.\begin{array}{r} \text{Find } \mathbf{x} \\ \text{such that} \\ \mathbf{c}'\mathbf{x} + \frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x} \gtrsim z_0, \\ \mathbf{A}_i\mathbf{x} \lesssim b_i,\ i \in \mathbb{N}_m \\ \mathbf{x} \geq 0, \end{array}\right]$$

where $z_0 \in \mathbb{R}$ is the aspiration level of the DM and $p_0,\ p_i,\ i \in \mathbb{N}_m$ the tolerances for the objective and the set constraints, respectively. The membership function for the objective is defined by

$$\mu_0(z) = \begin{cases} 1,\ \ z < z_0, \\ \frac{(z_0+p_0)-z}{p_0},\ z \in [z_0, \leq z_0 + p_0] \\ 0,\ z \geq z_0 + p_0, \end{cases}$$

The membership function for the ith ($i \in \mathbb{N}_m$) constraint is also defined by

$$\mu_i(\mathbf{A}_i\mathbf{x}) = \begin{cases} 1,\ \ \mathbf{A}_i\mathbf{x} < b_i, \\ \frac{(b_i+p_i)-\mathbf{A}_i\mathbf{x}}{p_i},\ \mathbf{A}_i\mathbf{x} \in [b_i, b_i + p_i] \\ 0,\ \mathbf{A}_i\mathbf{x} \geq b_i + p_i. \end{cases}$$

An optimal solution is obtained by solving the crisp equivalent QP problem

$$\left.\begin{array}{r} \text{Find } \alpha \\ \text{such that} \\ \mathbf{c}'\mathbf{x} + \frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x} + \alpha p_0 \leq z_0 + p_0, \\ \mathbf{A}_i\mathbf{x} + \alpha p_i \leq b_i + p_i,\ i \in \mathbb{N}_m \\ \mathbf{x} \geq 0,\ \alpha \in [0, 1]. \end{array}\right]$$

D.3 Multiplier method

Let a nonlinear programming problem be defined as in Varian [21]

$$\min_{\mathbf{x}} f(\mathbf{x}) \text{ subject to } \mathbf{g}(\mathbf{x}) = 0 \text{ and } \mathbf{h}(\mathbf{x}) \leq 0,$$

where $\mathbf{g},\ \mathbf{h}$ are nonlinear vectorial functions and $\mathbf{x}$ a vector of variables. The multiplier method is based on the Uzawa algorithm, which is a dual gradient ascent algorithm. [18] The principle of the method may be described by the three steps:
i) predict the multipliers $\mathbf{p}^{(k)}$ and $\mathbf{q}^{(k)}$ that are associated with the constraints $\mathbf{g}(\mathbf{x}) = 0$ and $\mathbf{h}(\mathbf{x}) \leq 0$,
ii) then, minimize $f(\mathbf{x}) + \mathbf{p}^{(k)}\mathbf{g}(\mathbf{x}) + \mathbf{q}^{(k)}\mathbf{h}(\mathbf{x})$,
iii) then, update until to convergence as $\mathbf{p}^{(k+1)} = \mathbf{p}^{(k)} + c_1\mathbf{g}(\mathbf{x}^{(k)})$ and $\mathbf{q}^{(k+1)} = \mathbf{q}^{(k)} + c_2 \max\{0, \mathbf{h}(\mathbf{x}^{(k)})\}$, where the numbers $c_i, i = 1, 2$ are positive.

D.4 Numerical example

The following numerical example is taken from Bector and Chandra [1], pages 77-78. The fuzzy symmet-

[18] The multiplier method (also called augmented Lagrangian method) package of the software *MATHEMATICA* (see Varian [21]) uses the primitive $MultiplierMethod[f, \mathbf{g}, \mathbf{h}, \mathbf{x}, \mathbf{x0}, DualParameter \to True]$. This primitive is finding a local solution to a minimization problem where f is the criterion to be minimized, $\mathbf{g}$ a list (possibly empty) of equality constraints, $\mathbf{h}$ a list (possibly empty) of inequality constraints of the form $\mathbf{h}(\mathbf{x}) \leq 0$, $\mathbf{x}$ the list of variables and $\mathbf{x0}$ the initial conditions for $\mathbf{x}$. It returns the list of results $\left\{f^*, \{x1 \to x_1^*, \ldots\}\right\}$. The option DualParameter is providing information on feasibility and Lagrange and/or KKT multipliers.

ric QP problem is

$$\left.\begin{array}{r} \text{Find } (x_1, x_2) \\ \text{such that} \\ 2x_1 + x_2 + 4x_1^2 + 4x_1x_2 + 2x_2^2 \lesssim 51.88, \\ 4x_1 + 5x_2 \gtrsim 20, \\ 5x_1 + 4x_2 \gtrsim 20, \\ x_1 + x_2 \lesssim 30, \\ x_1,\ x_2 \geq 0. \end{array}\right]$$

Let the tolerances be $p_0 = 2.12,\ p_1 = 2,\ p_2 = 1,\ p_3 = 3$, the equivalent crisp QP problem is

$$\left.\begin{array}{r} \max \alpha \\ \text{subject to} \\ 2x_1 + x_2 + 4x_1^2 + 4x_1x_2 + 2x_2^2 + 2.12\alpha \leq 54, \\ 4x_1 + 5x_2 - 2\alpha \geq 18, \\ 5x_1 + 4x_2 - \alpha \geq 19, \\ x_1 + x_2 + 3\alpha \leq 33, \\ x_1,\ x_2 \geq 0, \\ \alpha \in [0, 1]. \end{array}\right]$$

The optimum solution of the QP problem, given by the multiplier method is $x_1^* = .9918,\ x_2^* = 3.7253,\ \alpha^* = .8599$. This result tells that the solution is obtained with a satisfaction level of 86 per cent.

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