Séminaire des dynamiques économiques complexes
Université de Lille 1: EQUIPPE-Labo PAINLEVE- CLERSE
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Systèmes différentiels à retards en avenir incertain: le modèle d'évolution stochastique de Lotka-Volterra

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contenu

- 1. La modélisation stochastique des phénomènes de population

  (La dynamique générale des interactions entre deux espèces; le cas particulier du système de Lotka-Volterra (LV); les étapes de la modélisation stochastique d'évènements; le modèle différentiel stochastique);

- 2. Le système stochastique sans retard de Lotka-Volterra

  (Le système différentiel stochastique (SDS) du modèle sans surpeuplement des espèces ; les trajectoires bruitées du système par MatLab; le SDS du modèle avec surpeuplement des espèces; les conséquences du bruit sur la trajectoire en cas de surpeuplement des proies par MatLab);

- 3. Le système stochastique à retards de Lotka-Volterra

  (Le système LV à retards généralisé à plus de deux espèces; le modèle LV à paramètres stochastiques; la probabilisation généralisée à tous les paramètres; la stabilité globale asymptotique en probabilité un).

1.A Formulation for 2 and more species

natural growth — interacting coefficient — overcrowding coefficient

$$\begin{cases} \dfrac{dx_1}{dt} = x_1\left(a_1 + b_{11}x_1 + b_{12}x_2\right) & \text{... biomass 1} \\ \dfrac{dx_2}{dt} = x_2\left(a_2 + b_{21}x_1 + b_{22}x_2\right) & \text{... biomass 2} \end{cases}$$

generalisation to n species

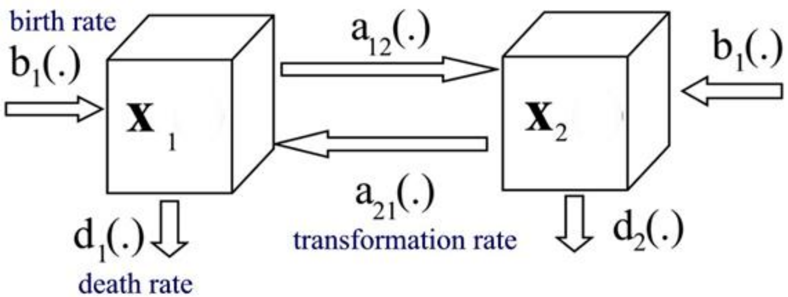
interaction rates — intrinsic growth rates

$$\frac{dx_i}{dt} = x_i\left(a_i + \sum_{j=1}^{n} b_{ij}x_j\right), \quad i = 1, \ldots, n$$

or

$$\frac{d\mathbf{x}}{dt} = \mathrm{diag}\left(\mathbf{x}(t)\right)\left(\mathbf{a} + \mathbf{B}\,\mathbf{x}(t)\right), \quad \mathbf{x}, \mathbf{a} \in \mathbb{R}^n, \mathbf{B} \in \mathbb{R}^{n\times n}$$

1.B General population system



$$\begin{cases} \dfrac{dx_1}{dt} = x_1\left(b_1\left(t, x_1, x_2\right) - d_1\left(t, x_1, x_2\right)\right) \\ \dfrac{dx_2}{dt} = x_2\left(b_2\left(t, x_1, x_2\right) - d_2\left(t, x_1, x_2\right)\right) \end{cases}$$

$$\begin{cases} \dfrac{dx_1}{dt} = x_1\left(b_1 - c_1 x_2\right) \\ \dfrac{dx_2}{dt} = x_2\left(c_2 x_1 - d_2\right) \end{cases}$$

Nonzero steady state equilibrium at :

$$\left(\frac{d_2}{c_2}, \frac{b_1}{c_1}\right)$$

Integral solutions are closed curves in the $(x_1 - x_2)$ plane to the steady state equilibrium.

1. La modélisation stochastique des phénomènes de population

1C. Steps of the stochastic modeling process :

step 1: population events and probabilities

- independent events (births, deaths and transformations): one event in each infenitesimal time intervals (i.e. multiple events are neglected)
- line 1: +1 birth in the population 1; the corresponding probability p1 is proportional to the density x1.
- line 5: one member of the population 1 changes his characteristic (the original population 1 changes by -1) into that of the destination population 2 (population 2 changes by +1); the corresponding probability is proportional to the density of the original population.

Change	Probability	Event
$\Delta \mathbf{x}^{(1)} = (1,0)^T$	$p_1 = b_1 x_1 \Delta t$	birth in x_1
$\Delta \mathbf{x}^{(2)} = (0,1)^T$	$p_2 = b_2 x_2 \Delta t$	birth in x_2
$\Delta \mathbf{x}^{(3)} = (-1,0)^T$	$p_3 = d_1 x_1 \Delta t$	death in x_1
$\Delta \mathbf{x}^{(4)} = (0,-1)^T$	$p_4 = d_2 x_2 \Delta t$	death in x_2
$\Delta \mathbf{x}^{(5)} = (-1,1)^T$	$p_5 = a_{12} x_1 \Delta t$	x_1 into x_2
$\Delta \mathbf{x}^{(6)} = (1,-1)^T$	$p_6 = a_{21} x_2 \Delta t$	x_2 into x_1
$\Delta \mathbf{x}^{(7)} = (0,0)^T$	$p_7 = 1 - \sum_{i=1}^{6} p_i$	no change

1C. Steps of the stochastic modeling process :

<u>step 2:</u> expectation vector of and covariance matrix

- find the mean change expectation and the covariance matrix V of the population changes for an infenitesimal time interval.
- the 2x2 symmetric matrix V, which is positive definite, has a positive square root B (involved by the modeling procedure)

Expectation vector of the population changes

$$E\left[\Delta \mathbf{x}\right] = \sum_{i=1}^{7} p_i \Delta \mathbf{x}^{(i)}$$

$$= \begin{pmatrix} b_1 x_1 - d_1 x_1 - a_{12} x_1 + a_{21} x_2 \\ b_2 x_2 - d_2 x_2 + a_{12} x_1 - a_{21} x_2 \end{pmatrix} \Delta t$$

Covariance matrix of the population changes

$$E\left[\Delta \mathbf{x}.\Delta \mathbf{x}^T\right] = \sum_{i=1}^{7} p_i \Delta \mathbf{x}^{(i)}.\Delta \mathbf{x}^{(i)T},$$

$$= \begin{pmatrix} b_1 x_1 + d_1 x_1 + \delta & -\delta \\ -\delta & b_2 x_2 + d_2 x_2 + \delta \end{pmatrix} \Delta t,$$

.

1C. Steps of the stochastic modeling process :

step 3: stochastic differential system

- several methods for the square root matrix determination: 1) Allen's formula for a 2x2 matrix V; 2) diagonalization of the nxn symmetric matrix V; 3) direct numerical procedure by Lakic & Petkovic (1998)

Determination of the squared root matrix B calculation

For $\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, we obtain $\mathbf{B} = \mathbf{V}^{1/2} = \frac{1}{d}\begin{pmatrix} a+w & b \\ b & c+w \end{pmatrix}$

where $w = \sqrt{ac - b^2}$ and $d = \sqrt{a + c + 2w}$ with $a = b_1x_1 + d_1x_1 + a_{12}x_1 + a_{21}x_2$, $b = -a_{12}x_1 - a_{21}x_2$ and $c = b_2x_2 + d_2x_2 + a_{12}x_1 + a_{21}x_2$.

Itô stochastic differential equation

$$d\mathbf{x}(t) = \boldsymbol{\mu}(t, x_1, x_2)\,dt + \mathbf{B}(t, x_1, x_2)\,d\mathbf{W}(t),$$

with $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{W}(t) = (W_1(t), W_2(t))^T$ is a two-dimensional Wiener process. From the general model, we deduce

$$\boldsymbol{\mu} = (-a_{12}x_1 + a_{21}x_2,\, a_{12}x_1 - a_{21}x_2)^T,$$

$$\mathbf{V} = \begin{pmatrix} a_{12}x_1 + a_{21}x_2 & -a_{12}x_1 - a_{21}x_2 \\ -a_{12}x_1 - a_{21}x_2 & a_{12}x_1 + a_{21}x_2 \end{pmatrix},$$

and

$$\mathbf{B} = \mathbf{V}^{1/2} = \sqrt{\frac{a_{12}x_1 + a_{21}x_2}{2}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

2. Le système stochastique sans retard de Lotka-Volterra

2.A Stochastic Lotka-Volterra system without delays : variant 1: the PP system without overcrowding

- a competion system between two species without internal competition (no overcrowding for low densities of the populations);
- the mean vector is the drift of the SDE, and square root matrix B is the diffusion of the process

Stochastic LV system without overcrowding

The standard PP model is obtained for $b_1(.) = b_1$, $d_1(.) = c_1 x_2$, $b_2(.) = c_2 x_1$ and $d_2(.) = d_2$. In the stochastic version of the model, since we have $a_{12} = a_{21} = 0$, the mean vector is

$$\boldsymbol{\mu} = \begin{pmatrix} (b_1 - c_1 x_2)\, x_1 \\ (c_2 x_1 - d_2)\, x_2 \end{pmatrix}$$

and the diffusion matrix is

$$\mathbf{B} = \mathbf{V}^{1/2} = \begin{pmatrix} \sqrt{(b_1 + c_1 x_2)\, x_1} & 0 \\ 0 & \sqrt{(c_2 x_1 + d_2)\, x_2} \end{pmatrix}$$

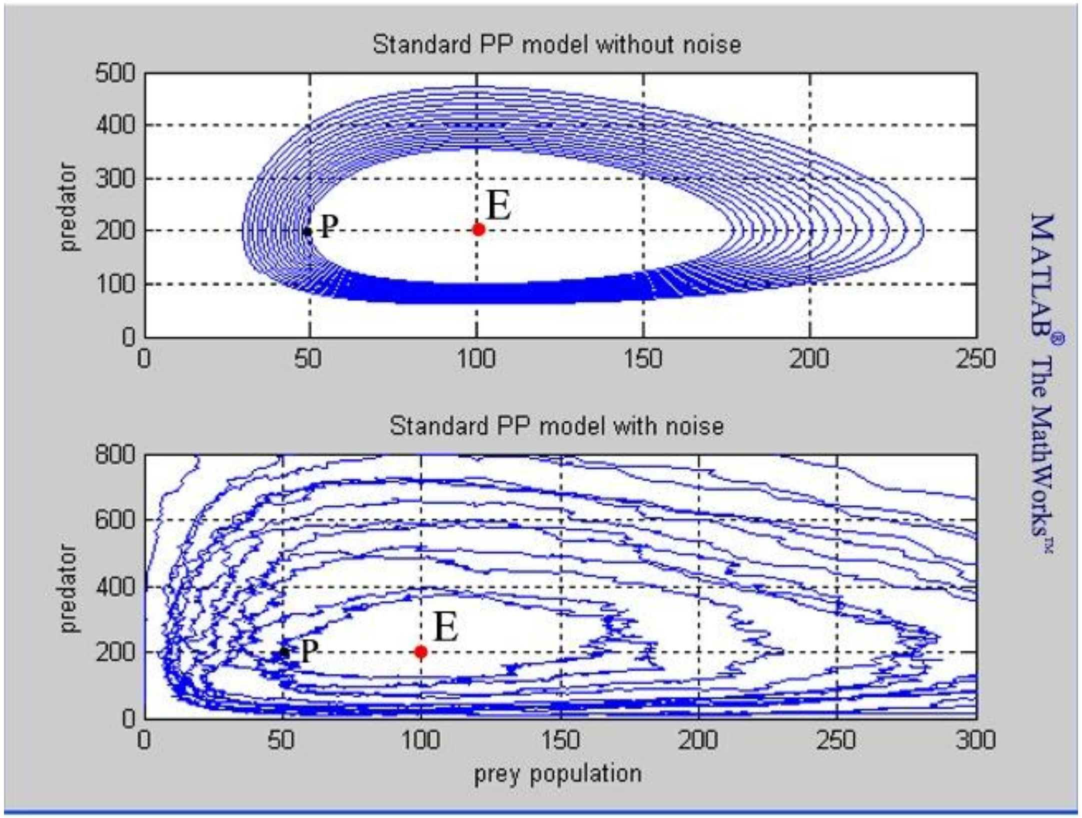
Omitting the time argument, the standard stochastic PP model takes the form

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} (b_1 - c_1 x_2)\, x_1 \\ (c_2 x_1 - d_2)\, x_2 \end{pmatrix} dt + \begin{pmatrix} \sqrt{(b_1 + c_1 x_2)\, x_1} & 0 \\ 0 & \sqrt{(c_2 x_1 + d_2)\, x_2} \end{pmatrix} \begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix}.$$

2.A Stochastic Lotka-Volterra system without delays :

variant 1: the PP system without overcrowding (followed)

- the parameter values of the numerical application: b1=d2=2, c1=0.01,c2=0.002.
- the figures picture the deterministic and stochastic trajectories.
- both figures have a cyclical pattern around the nonzero equilibrium E(100,200).
- the noise perturbes the trajectories.



2.B Stochastic Lotka-Volterra system without delays : variant 2: the PP system with overcrowding

- Preys and predators come into competition with themselves, as a consequence of a population overcrowding phenomena (e.g;. Food limitations)
- This phenomena introduces new factors (beta times density) into the system
- The stochastic differential matrix equation introduce the new parameter sigma, which scales the noise.

Stochastic LV system with overcrowding

We have the deterministic system

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)\left(b_1 - c_1 x_2(t) - \beta x_1(t)\right) \\ \dfrac{dx_2(t)}{dt} = x_2(t)\left(-d_2 + c_2 x_1(t) - \delta x_2(t)\right). \end{cases}$$

The stochastic version of the model introduces a multiplicative noise and takes the form (omitting the time argument)

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} \left(b_1 - c_1 x_2 - \beta x_1\right) x_1 \\ \left(-d_2 + c_2 x_1 - \delta x_2\right) x_2 \end{pmatrix} dt + \begin{pmatrix} \sigma x_1 & 0 \\ 0 & \sigma x_2 \end{pmatrix} \begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix},$$

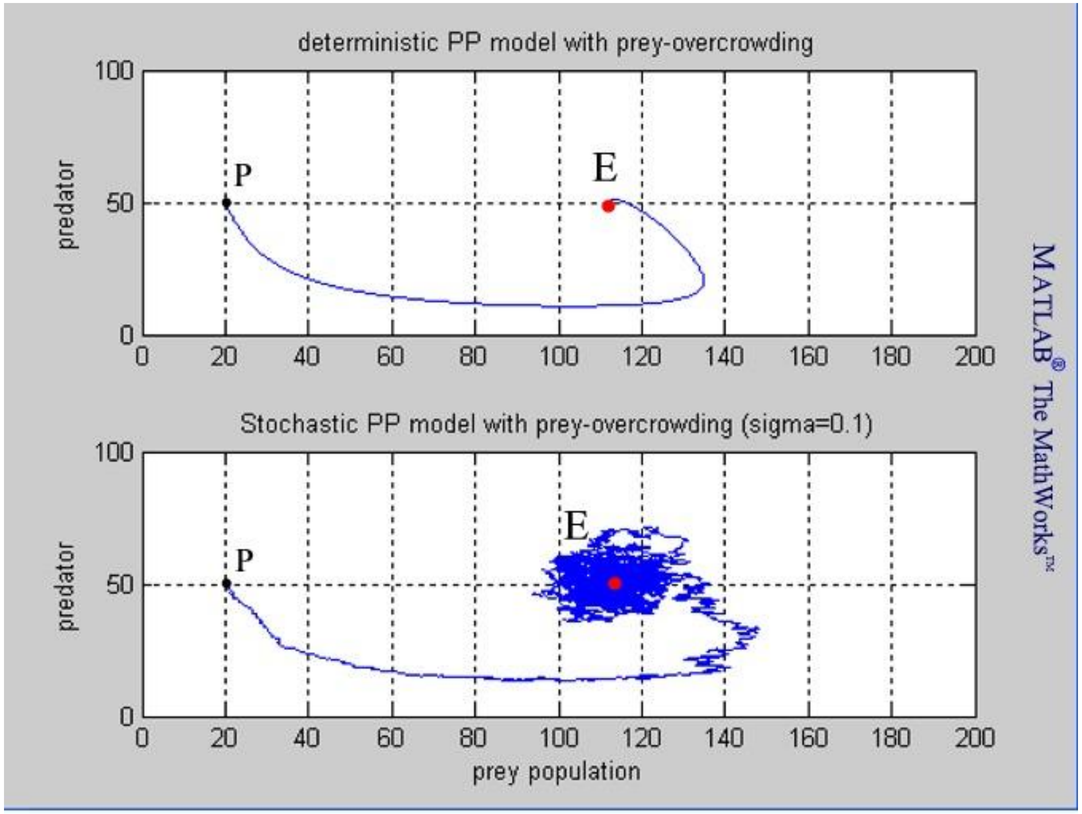
where σ scales the amplitude of the noise .

2.B Stochastic Lotka-Volterra system without delays :

variant 2: the PP system with overcrowding (followed)

- the PP model is prey-overcrowded
- the parameter values of the numerical application: b1=d2=2, c1=0.01,c2=0.002, beta=0.0133

- the figures picture the globally asymptotically convergent deterministic and stochastic trajectory.
- both figures have a spiral pattern directed towards the nonzero equilibrium E(112.5,50).
- the noise perturbes the trajectories.



3. Le système stochastique à retards de Lotka-Volterra

3A. Generalized stochastic system with delays : outline

- the LV differential system with delays is generalized to n species
- In a noisy environment , only the intrinsic growth rates are perturbed.
- a stochastic differential matrix equation is obtained

The LV differential system with delays

A delayed effect of one species[1] on another is introduced by means of lagged interaction terms, such as

$$\frac{d\mathbf{x}(t)}{dt} = \mathrm{diag}\left(\mathbf{x}(t)\right)\left(\mathbf{a} + \mathbf{B}\,\mathbf{x}(t-\tau)\right),$$

where $\mathbf{x},\mathbf{a} \in \mathbb{R}^n$, $\mathbf{B} \in \mathbb{R}^{n\times n}$. The autonomous competitive or cooperative L-V system may have several time-delays,

$$\frac{dx_i(t)}{dt} = x_i(t)\left(b_i - \sum_{j=1}^{n} a_{ij}x_j(t) - \sum_{j=1}^{n} b_{ij}x_j(t-\tau_{ij})\right),\ i = 1,\ldots,n$$

where $i = 1,\ldots,n$. All the coefficients are real constants deterministic

The generalized LV differential system with delays

The following delay LV system generalizes the preceding deterministic n-dimensional system . We have ,

$$\frac{d\mathbf{x}(t)}{dt} = \mathrm{diag}\left(\mathbf{x}(t)\right)\left(\mathbf{b} + \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau)\right),\ \mathbf{x} \in \mathbb{R}^n$$

Suppose a noisy environment, where the intrinsic growth rates b_i 's are replaced by

$b_i + \sigma_{ii}\left(x_j - \bar{x}_j\right)dW(t)/dt$, where $\bar{x}_j$ is an equilibrium state component, σ_{ii} 's positive constants,

$W(t)$ a Brownian motion on a completely probability space $\left(\Omega, \mathcal{F}, \left\{\mathcal{F}_t\right\}_{t\geq 0}, \mathbb{P}\right)$. The corresponding Lotka-Volterra SDDE is

$$d\mathbf{x}(t) = \mathrm{diag}\left(\mathbf{x}(t)\right)\left\{\left(\mathbf{A}(\mathbf{x}(t)-\bar{\mathbf{x}}) + \mathbf{B}(\mathbf{x}(t-\tau)-\bar{\mathbf{x}})\right)dt + \boldsymbol{\sigma}(\mathbf{x}(t)-\bar{\mathbf{x}})\,d\mathbf{W}\right\}$$

3A. Generalized stochastic system with delays : properties

Properties of the generalized stochastic LV system with delays

Assumption. The noise intensity matrix $\boldsymbol{\sigma} = \left(\sigma_{ij}\right)_{n\times n}$ supposes that (H1): $\sigma_{ii} > 0$ if $1 \le i \le n$, while $\sigma_{ij} \ge 0$ if $i \ne j$.

Theorem (Mao, Yuan, & Zou, 2005) *Under assumption (H1), for any coefficients* $\mathbf{A}$, $\mathbf{B}$ *and any initial data* $\left\{\mathbf{x}(t) : t \in [-\tau, 0]\right\} \in C\left([-\tau, 0]; \mathbb{R}_+^n\right)$, *there is a unique global solution* $\mathbf{x}(t)$ *to the system on* $t \ge -\tau$. *Moreover, the solution will remain in the cone* $\mathbb{R}_+^n$ *with probability one.*

3B. Application : the food chain

- one prey population and two predator populations (one is intermediate and the other is superior)
- for this application, two conditions are to be satisfied to have a globally asymptotically stable equilibrium with the probability one.

The stochastic version of the PP model , around the equilibrium state $\bar{\mathbf{x}}$ is

$$d\mathbf{x}(t) = \mathrm{diag}(\mathbf{x}(t))\{(\mathbf{A}(\mathbf{x}(t)-\bar{\mathbf{x}}) + \mathbf{B}(\mathbf{x}(t-\tau)-\bar{\mathbf{x}}))dt + \boldsymbol{\sigma}(\mathbf{x}(t-\tau)-\bar{\mathbf{x}})d\mathbf{W}(t)\}$$

where $\mathbf{x} \in \mathbb{R}^3$ and $\boldsymbol{\sigma} = \mathrm{diag}(\sigma_{11}, \sigma_{22}, \sigma_{33})$

and

$$\mathbf{b} = \begin{pmatrix} b_1 \\ -b_2 \\ -b_3 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -a_{11} & 0 & 0 \\ 0 & -a_{22} & 0 \\ 0 & 0 & -a_{33} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & -a_{12} & 0 \\ a_{21} & 0 & -a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}.$$

The states x_1, x_2 and x_3 are respectively the population densities for a prey, an intermediate predator and a top predator. A stationary equilibrium $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)^T$ exists in the positive cone $\mathbb{R}^3_+$, if

$$b_1 - \frac{a_{11}}{a_{21}} b_2 - \frac{a_{11}a_{22} + a_{12}a_{21}}{a_{21}a_{32}} b_3 > 0$$

Mao *et al* (2005) conclude that the equilibrium $\bar{\mathbf{x}}$ is globally asymptotically stable with probability one , if two conditions are satisfied. Letting $\hat{c} = a_{11}^{-2} + a_{22}^{-2} + a_{33}^{-2}$, we have the two conditions

$$\hat{c}\{(a_{12}^2 + a_{32}^2) \vee (a_{21}^2 + a_{23}^2)\} \leq 1$$

and

$$\sigma_{ii}^2 \leq \frac{a_{ii}}{\bar{x}_i}\{1 - \hat{c}((a_{12}^2 + a_{32}^2) \vee (a_{21}^2 + a_{23}^2))\}, 1 \leq i \leq n$$

Fin de la présentation

Merci pour votre présence