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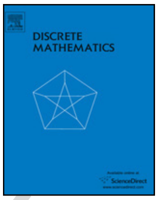

Graph theory and economic models: From small- to large-size applications

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ABSTRACT

This empirical study explores the structure of macroeconomic models using major concepts and algorithms of the graph theory. Different sizes of applications with dynamic effects are being considered. We will first examine the bipartite matching problem when assigning the variables to the equations. We will also propose a simple method for improving the regular circular embedding of graphs on the basis of one of the longest circuits by permutating the vertices. The distribution of all the circuits is shown according to their length. The determination of the maximal list of edge-disjoint circuits also produces a useful insight into the structure. A typology of the interdependent variables is proposed using the properties of one all-pairs shortest paths matrix. This classification is based on both the diffusion effects of points towards the rest of the graph and the perturbations that the rest of the graph exerts on these points.

1. Introduction

This empirical study explores the structure of macroeconomic models using major concepts and algorithms of the graph theory [7]. Different sizes of applications with dynamic effects are being considered. Two models of the same application for the Netherlands economy are considered in this paper: the small-size industry sub-model of the CS (Conjunctural-Structural) model with 7 equations, and the complete large-size CS model with 82 equations. The CS sub-model is first introduced to present the retained methodology: building the directed graph (or digraph) from the set of equations (matching problem, circular embedding), determining the directed acyclic graph (DAG), looking for the set of all elementary circuits and those of the non-edge-disjoint circuits, considering a typology of the vertices based on the properties of the graph. The complete CS model is thereafter analyzed according to a comparable approach: associated digraphs to the static and dynamic versions of the model, the DAGs, all circuits enumerations and typologies. The computer calculations have been effected using the software Mathematica® 5.1 and its specialized packages DecisionAnalysis’Combinatorica, ’GraphPlot. These packages can be found at http://library.wolfram.com/infocenter/ (see also [10–12]).

1.1. *Description of the model*

Let us introduce a small-size macroeconomic model that will be the basic reference for further applications. The yearly CS industry sub-model in [2,1] for Netherlands is described by equations, where most of the variables are expressed in percentage change.[1] The set of variables is composed of endogenous and exogenous variables. The variables are instantaneous and may be delayed by several periods (years). We have a system of seven equations

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[1] The variables on population and unemployment are in numbers (× 100,000). A dot above a symbol refers to percentage changes (divided by 100), such that $\dot{l} = (l - l_{-1})/l_{-1}$, where l_{-1} is the value of l at the previous time period. The variable $\dot{p}_{c_{-1}}$ is delayed by one year, $\dot{p}_{c_{-2}}$ by two years. A variable $h_{-\frac{1}{2}}$ is $(h + h_{-1})/2$, and $\Delta q = q - q_{-1}$.

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$$\dot{l} = 0.086 + 0.22\left(\dot{p_c} + \dot{p}_{c_{-1}} + \dot{p}_{c_{-2}}\right) + 0.79\dot{h}_{-\frac{1}{2}} - 0.09w_{-\frac{1}{2}} + \dot{l}_{au}, \tag{1}$$

$$\dot{p_c} = 0.006 + 0.5(\dot{l} - \dot{h})_{-\frac{1}{2}} + 0.25\dot{p}_{mgr} + 0.072\dot{p}_{mc} + \dot{p}_{cau} - \left(\frac{n^*_{b_{-1}}}{v_{-1}} - \frac{n^*_{b_{-2}}}{v_{-2}}\right), \tag{2}$$

$$\dot{h} = \frac{y}{y_{-1}} \div \frac{a_b}{a_{b-1}} - 1, \tag{3}$$

$$\dot{a_b} = \dot{c^*_p} - 0.4\Delta q + \Delta a_{bat\,v} - 0.033 - 0.03\sum_{i=0}^{-5}\left(\frac{1+\dot{l}}{1-\dot{p}_{i_{-wo}}} - 1\right), \tag{4}$$

$$a_b = a_{b_{-1}} + \Delta a_b, \tag{5}$$

$$a_l = a_{l_{-1}} + \Delta a_b - \Delta a_z, \tag{6}$$

$$\Delta L_b = \left(\frac{a_l(\dot{l}+1)}{a_{l_{-1}}} - 1\right)L_{b_{-1}}. \tag{7}$$

The endogenous variables are: a_b the employment, a_l the dependent employment, h the labor productivity, l the wage level, L_b the disposable wage income, and p_c the price level of private consumption. The exogenous variables are: $a_{bat\,v}$ the employment induced by change in working hours, a_z the self-employment working population, c^*_p the productive capacity adjusted for autonomous influences, l_{au} the autonomous wage level, p_{cau} the autonomous price level of private consumption, $p_{i_{-wo}}$ the price level of imported goods (excluding housing), n^*_b/v the ratio of stock (excluding live stock) to the total sales, p_{mc} the price level of imported goods, p_{mgr} the price level of imported raw material, and y the gross industrial product.

Eqs. (1), (2) and (4) are reaction functions, the other relations being definitions. The CS sub-model is concentrated on the determination of wages, prices, labor productivity and employment in industries. Eq. (1) is a Phillips–Lipsey function where wage rates are depending on prices (with lags over a period of two years), unemployment and labor productivity (with a lag of six months). Eq. (2) is a price equation of usual type, where prices of expenditures are explained by wage costs, labor productivity and import prices. Prices are also influenced by stock positions. Eq. (4) is for labor demand by industries. Labor demand depends on production capacity, real wage costs (with lags over a period of 5 years) and a technological trend. In this equation, real wages account for factor substitution.

1.2. A classical resolution

A static version of the sub-model is obtained as in [2] replacing all the predetermined variables by their observed values (yearly 1957 figures). A normalized system will be obtained[2] in which the variable in the LHS of each equation is calculated by the variables of the RHS. We have

$$\dot{l} = 0.087 + 0.22\dot{p_c} + 0.395\dot{h}, \tag{8}$$

$$\dot{p_c} = 0.04 + 0.25(\dot{l} - \dot{h}), \tag{9}$$

$$\dot{h} = \frac{37.26}{a_b} - 1, \tag{10}$$

$$\Delta a_b = 0.297 - 0.1036\dot{l}, \tag{11}$$

$$a_b = 36.02 + \Delta a_b \tag{12}$$

$$a_l = 26.69 + \Delta a_b \tag{13}$$

$$\Delta L_b = -11.9 + 0.45a_l(1 + \dot{l}). \tag{14}$$

Fig. 1(a) shows the interactions between variables. the resulting graph consists of 7 vertices and 10 oriented edges (or arcs) whose $(\dot{p}_c\dot{l})$ arc is oriented in both directions. Let us use the classic solving method for electrical networks. At the first step, the circuit $\{\dot{l}, \Delta a_b, a_b, \dot{h}, \dot{l}\}$ of the Fig. 1(a) is replaced by a single node λ, called the auxiliary variable. A sub-system of the four Eqs. (8) and (10)–(12) is associated to that circuit. If the system is solvable, the variables $\dot{l}$, $\dot{h}$, Δa_b, a_b are expressions of the remaining variables which are considered as parameters. A necessary and sufficient condition for a solution tells that the Jacobian determinant $|J|$ is non-zero. We have

$$J = \begin{pmatrix} 1 & 0 & 0 & -0.395 \\ 0.1036 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 37.26a_b^{-2} & 1 \end{pmatrix}.$$

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[2] The normalized system tells us which equation will calculate each endogenous variable of the model. This form is generally deduced from the economic theory and may be not unique.

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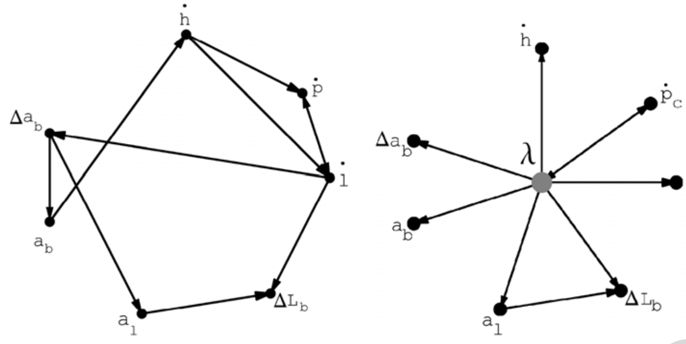

(a) Initial graph. (b) Step 1.

Fig. 1. Classical solving method (first step).

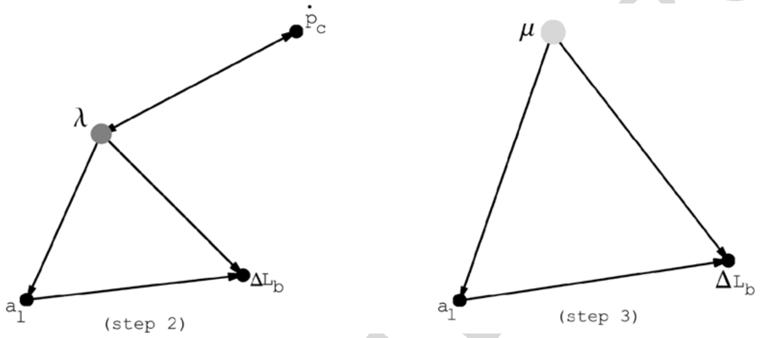

Fig. 2. Classical solving method (steps two and three).

The solutions are[3]

$$\left\{\dot{l} = f_1(\dot{p}_c, \lambda), \dot{h} = f_2(\dot{p}_c, \lambda), \Delta a_b = f_3(\dot{p}_c, \lambda), a_b = f_4(\dot{p}_c, \lambda)\right\},$$

where the auxiliary variable λ takes positive integer values that correspond to the number of solutions. After successive eliminations of the variables $\dot{h}$, a_b and Δa_b, the following expression of $\dot{l}$ has been obtained

$$\dot{l} = -.308 + \frac{14.7177}{36.317 - 0.1036\dot{l}} + 0.22\dot{p}_c. \tag{15}$$

The expression (15) takes also the form

$$3.53206 + 0.1036(\dot{l})^2 - \dot{l}(36.2851 + 0.022792\dot{p}_c) + 7.98974\dot{p}_c = 0.$$

Two solutions[4] are obtained with $\lambda = 1, 2$

$$\dot{l}' = 175.121 - 0.11\sqrt{-1703.16 + \dot{p}_c}\sqrt{-1486.46 + \dot{p}_c} + 0.11\dot{p}_c,$$

and

$$\dot{l}'' = 175.121 + 0.11\sqrt{-1703.16 + \dot{p}_c}\sqrt{-1486.46 + \dot{p}_c} + 0.11\dot{p}_c.$$

This process of contracting[5] is illustrated in Figs. 1 and 2. The unique solution is given by

$$\begin{aligned} &\dot{l} = 0.1108, \quad \dot{p}_c = 0.0611, \quad \dot{h} = 0.0263, \quad \Delta a_b = 0.2855, \\ &a_b = 36.3, \quad a_l = 26.9755, \quad \Delta L_b = 1.5844. \end{aligned}$$

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[3] According to the existence–uniqueness theorem, $|J|$ must be non-zero for a unique solution. Since $|J| = 1 - 1.52475 a_b^{-2}$, we must have $a_b \neq \sqrt{1.52475}$. This condition will always be satisfied for the variable a_b (employment in industries) whose value for 1957 differs significantly with about $36.3(10^5)$ of employees.

[4] At this stage of the resolution, two real solutions suppose that $\dot{p}_c \geq 1486.46$. This condition cannot be satisfied with the percentage changes of price.

[5] A semi-reduced form of the model can be achieved at an intermediate step of its resolution (see [6]). Parameterized supply and demand functions or IS-LM equilibrium equations are then deduced.

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$$M=\begin{array}{cc|ccccccc|}
 & & \multicolumn{7}{c}{\text{variable \#}} \\
 & & \dot{i} & \dot{p}_c & \dot{h} & \Delta a_b & a_b & a_1 & \Delta L_b \\
\text{equation \#} & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & 4 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
 & 5 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
 & 6 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

Fig. 3. A 7 × 7 matrix representation.

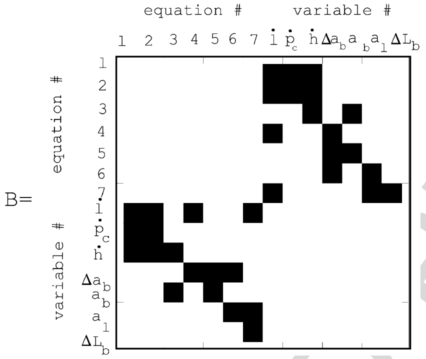

Fig. 4. Bi-adjacency 14 × 14 matrix.

2. The matching problem

In the former normalized form of the sub-model, the matching of the variables to the equations has been imposed. Let us consider the matching problem in general. In the following 0–1 matrix M of the CS sub-model, the rows state for the equations and the columns for the variables. The entries m_{ij} are such that

$$m_{ij}=\begin{cases}1 & \text{if a variable } \sharp \text{ in column } j \text{ is present in equation } \sharp \text{ in line } i,\\ 0 & \text{otherwise.}\end{cases}$$

The matrix M is shown in Fig. 3. To solve the model, one has to assign each variable to one single equation. This problem is known as a matching problem.

2.1. Presentation

Matching is a graph optimization problem (see [8]: 254–305). Let us introduce some definitions from [4,5].

Definition 1 (*Bipartite Graph*). A graph $G=(U\bigcup W,E)$ is bipartite if its set of vertices can be partitioned into two sets U and W, such that every edge in G has one endpoint in U and one endpoint in W. The sets U and W are the color classes of G and (U,W) a bipartition of G.

Definition 2 (*Bipartite and Perfect Matching*).

1. A bipartite matching $\mathcal{M}$ is a set of pairwise non-adjacent edges in a bipartite graph $G=(U\bigcup W,E)$ where U may denote the set of equations of a given system and W the set of variables. That is, $\mathcal{M}\subseteq E(G)$ such that $e_1,e_2\in\mathcal{M}$, $e_1=(i_1,j_1)$, $e_2=(i_2,j_2)$ and $i_1=i_2\Leftrightarrow j_1=j_2$.
2. A perfect matching $p(\mathcal{M})$ of the bipartite graph $G=(U\bigcup W,E)$ is a pairing of the set U to the set W which uses each element of U and each element of W, once and only once. Such a matching covers all the vertices of the graph.

The bi-adjacency matrix $B=(b_{ij})$ is defined by

$$b_{ij}=\begin{cases}1 & \text{if } (u_iw_j)\in E(G),\\ 0 & \text{otherwise.}\end{cases}$$

The bi-adjacency matrix of the CS model is shown in Fig. 4: for convenience, a black square states for one 1 and a point for zero. A matching matches each vertex in $U=\{u_1\dots u_n\}$ to one in $W=\{w_1,\dots,w_n\}$. Hall's marriage theorem (see [8]) states that there is a matching in which every equation can be married, if and only if, every subset S of equations knows a subset of variables at least as large as $|S|$. A polynomial-time matching algorithm follows from Berge's theorem, which states that a matching is maximum, if and only if, it contains no augmenting path. The algorithm starts with an arbitrary matching. This matching can be improved by finding an $\mathcal{M}$-augmenting path P, if any. Then $\mathcal{M}$ is replaced with the symmetric

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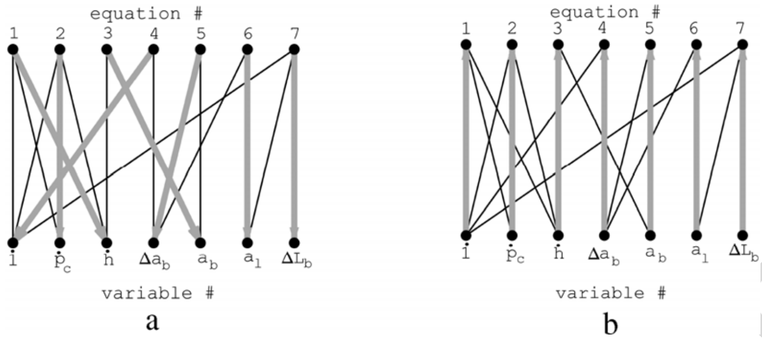

Fig. 5. Maximum matching and maximum restricted matching.

difference $(\mathcal{M} - P)\bigcup(P - \mathcal{M})$. The matching is maximum when it contains no augmenting path. Given a graph G with bipartition $V(G) = (U, W)$. Let us denote $\nabla(v)$ the set of edges incident to v. A 0–1 vector $\mathbf{x}$ in $\mathbb{R}^{E(G)}$ is the incidence vector of a matching, if and only if, $\mathbf{x}(\nabla(v)) \leq 1$ for every vertex $v \in V(G)$.

2.2. Solution

The linear programming problem in matrix form, is

$$\max_{\mathbf{x}} \mathbf{1} \cdot \mathbf{x}$$

$$\text{s.t. } \mathbf{A} \cdot \mathbf{x} \leq \mathbf{1}$$

$$\mathbf{x} \geq \mathbf{0},$$

where $\mathbf{1}$ is a vector of ones and $\mathbf{A} = (a_{ve})$ the incidence matrix of G with

$$a_{ve} = \begin{cases} 1 & \text{if the vertex } v \text{ is incident on the edge } e, \\ 0 & \text{otherwise.} \end{cases}$$

The set of the inequations forms a polytope $\mathcal{M}(G)$. The solutions are those which maximize the objective function $\mathbf{1} \cdot \mathbf{x}$. The incidence matrix of the CS sub-model is shown in Fig. 6.

The system of inequality constraints is

$$\begin{aligned}
&e_1 + e_2 + e_3 \leq 1, \quad e_1 \geq 0, e_2 \geq 0, e_3 \geq 0, \\
&e_4 + e_5 + e_6 \leq 1, \quad e_4 \geq 0, e_5 \geq 0, e_6 \geq 0, \\
&e_7 + e_8 \leq 1, \quad e_7 \geq 0, e_8 \geq 0, \\
&e_9 + e_{10} \leq 1, \quad e_9 \geq 0, e_{10} \geq 0, \\
&e_{11} + e_{12} \leq 1, \quad e_{11} \geq 0, e_{12} \geq 0, \\
&e_{13} + e_{14} \leq 1, \quad e_{13} \geq 0, e_{14} \geq 0, \\
&e_{15} + e_{16} + e_{17} \leq 1, \quad e_{15} \geq 0, e_{16} \geq 0, e_{17} \geq 0, \\
&e_1 + e_4 + e_9 + e_{15} \leq 1, \\
&e_2 + e_5 \leq 1, \\
&e_3 + e_6 + e_7 \leq 1, \\
&e_{10} + e_{11} + e_{13} \leq 1, \\
&e_8 + e_{12} \leq 1, \\
&e_{14} + e_{16} \leq 1, \\
&e_{17} \leq 1.
\end{aligned}$$

Among the optimal solutions of that system there will be a 0–1 vector.

Theorem 3 ([8]). *Let G be a bipartite graph. Then the vertices of the polytope $\{\mathbf{x} \in \mathbb{R}^{E(G)}, \mathbf{x} \geq 0, \mathbf{A} \cdot \mathbf{x} \leq \mathbf{1}\}$ are 0–1 vectors. In fact, they are exactly the incidence vectors of matchings.*

The computation renders a technical matching which is shown in Fig. 5(a). However, the economist would not accept this assignment of the variables to the equations. Indeed the econometric equations generally impose the causality going from the explanatory to the explained variables. Let us impose that the econometric equations (1), (2) and (4) compute the adequate variable. Then the variable $\dot{l}$ will be assigned to (1), variable $\dot{p}_c$ to (2) and variable $\dot{a}_b$ to (4). The matching is shown in Fig. 5(b) will then be retained by the economists.

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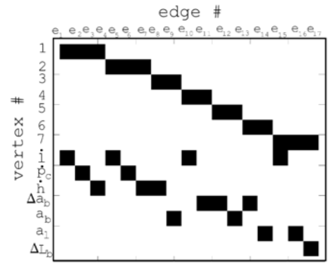

Fig. 6. Incidence matrix.

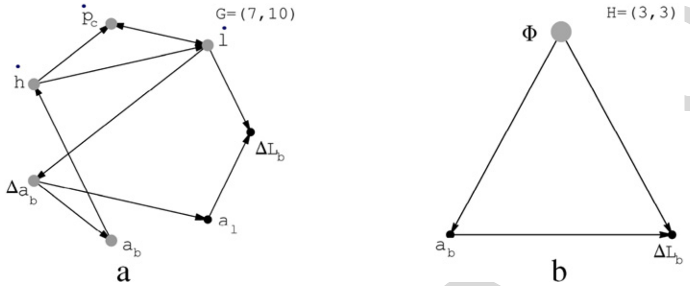

Fig. 7. Circular embedding of the graph and DAG.

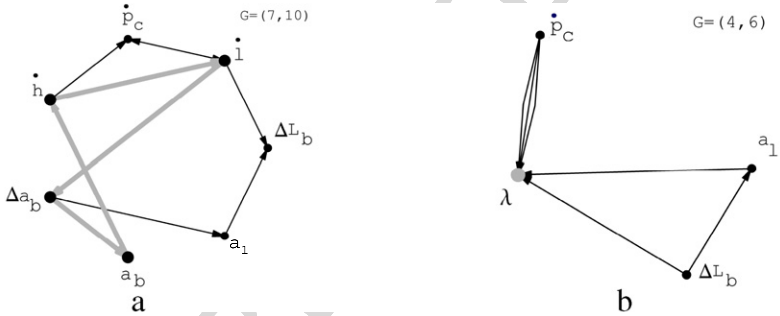

Fig. 8. Strong component (a) and contracted graph (b).

3. Embedding of the small-size graph

In a circular embedding the vertices are placed on the circumference of a unit circle, computing evenly spaced points. Considering that the property stating that no three vertices are collinear, an illustration is given in Fig. 7(a) for the directed graph $G = (7, 10)$, with 7 vertices and 10 oriented edges.[6]

Definition 4 (*Connected Component*). Connected components of a graph are maximal connected subgraphs.

Definition 5 (*Strong Component*). A strong component of a graph (SC) is a maximal strongly connected subgraph, where the vertex sets partition the set V and does not include all edges of G.

The strong component (SC) is shown in Fig. 7(b) as a set of gray points. The directed acyclic graph (DAG) is achieved after contracting these sets and reveals the structure of the graph.

Proposition 6. *The condensation of a digraph is an acyclic digraph DAG which contains no circuits. This operation may produce multiple parallel edges.*

Contracting a pair of vertices v_1 and v_2 replaces them by one vertex v such that it is adjacent to any adjacent vertex of v_1 and v_2.[7] The procedure of contracting a strong component is illustrated in Fig. 8.

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[6] This graph is simple since it has neither self-loops nor multi-arcs. It is a sparse graph since the cardinalities of sets V and E are close together.

[7] Let G be an n-vertex graph and a subset S of k vertices to contract. The resulting graph H has $n - k + 1$ vertices. Each vertex in S is mapped to the $n - k + 1$ vertices of H. Contract runs in linear time $\mathcal{O}(|V|)$.

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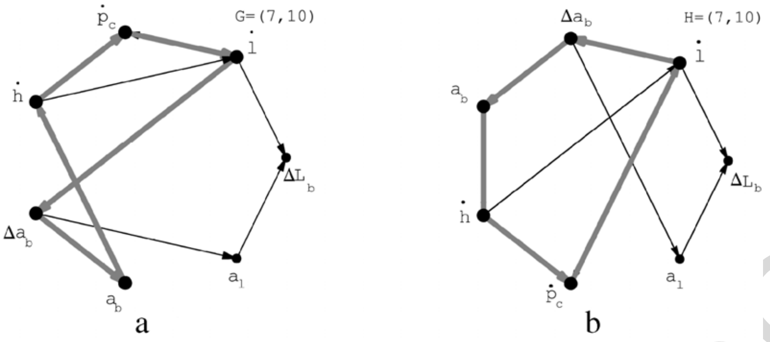
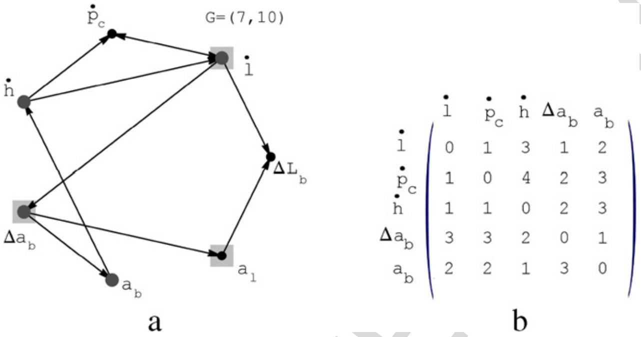


Fig. 9. Initial (a) and ordered (b) longest circuit.



$$\begin{array}{c|ccccc} & \dot{l} & \dot{p}_c & \dot{h} & \Delta a_b & a_b \\ \hline \dot{l} & 0 & 1 & 3 & 1 & 2 \\ \dot{p}_c & 1 & 0 & 4 & 2 & 3 \\ \dot{h} & 1 & 1 & 0 & 2 & 3 \\ \Delta a_b & 3 & 3 & 2 & 0 & 1 \\ a_b & 2 & 2 & 1 & 3 & 0 \end{array}$$

Fig. 10. Central vertices (big points), articulation points (gray squares) and all-pairs shortest paths.

4. Set of circuits and non-edge-disjoint circuits

The initial digraph of the sub-model consists in 3 circuits, defined by elementary closed paths like $C = \{v_1, v_2, \ldots, v_1\}$. The longest circuit is composed of the list $\{\dot{l}, \Delta a_b, a_b, \dot{h}, \dot{p}_c, \dot{l}\}$. Fig. 9(b) shows an improved representation where the set of vertices is decomposed into two subsets: the subset of reordered vertices of the longest circuit, and the subset of remaining vertices of the graph. Thus the graph H, obtained after successive permutations of its vertices, is isomorphic to G.[8] A maximal list of edge-disjoint cycles is shown in Fig. 9(b). The extracted edge-disjoint cycles are highlighted.

5. Invariant properties of a small-size model

5.1. All-pairs shortest paths matrix

The computation of the all-pairs shortest paths matrix is essential for judging the eccentricity of the SC. This squared matrix collects all finite distances $d(x, y)$ that are precisely the length of the shortest x–y paths (one path always exists in the SC where all vertices are reachable)[9] using Dijkstra's algorithm or the Bellman–Ford algorithm (see [10,9,3]). The all-pairs shortest paths matrix is shown in Fig. 10(b). Several graph invariants are depending on this distance matrix.

5.2. Eccentricity, central and peripheral vertices, articulation vertices

The eccentricity for each vertex v is simply the maximum of the shortest paths starting from v.

Definition 7 (*Eccentricity of a Vertex*)). The eccentricity of a vertex v in graph G $\mathtt{ecc}(v)$ is the distance from v to any farthest vertex from it. Hence, we have

$$\mathtt{ecc}(v) = \max_{x \in V(G)} \{d(v, x)\}.$$

In this example, we have the list {3, 4, 3, 3, 3}. Some graph invariants are deduced. The minimum eccentricity is the radius $\mathtt{rad}(G)$ and the maximum eccentricity the diameter $\mathtt{diam}(G)$.

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[8] There exists a bijection $\phi : V(G) \mapsto V(H)$ such as for every pair $u, v \in (G)$ one have an edge $(uv) \in E(G)$, if and only if, there is an edge $(\phi(u)\phi(v)) \in E(H)$.

[9] The all-pairs shortest paths can be computed in $\mathcal{O}(|V|^2.|E|)$ time using the Bellman–Ford algorithm or in $\mathcal{O}(|V|^3)$ time using Dijkstra's algorithm.

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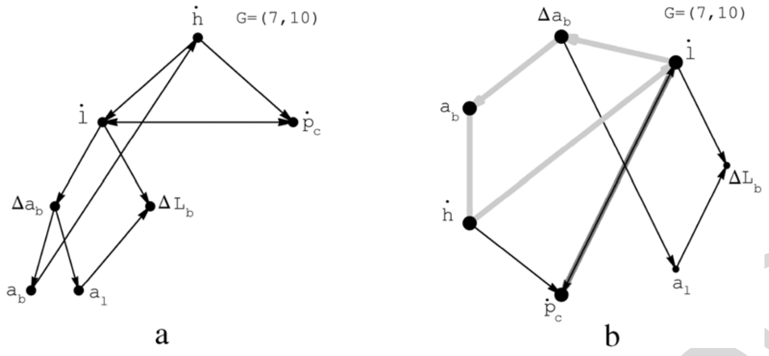

Fig. 11. Root embedded graph and non-edge-disjoint circuits.

Definition 8 (*Radius, Diameter of a Graph*).

1. The radius is the minimum eccentricity

$$\mathtt{rad}(G) = \min_{x \in V(G)} \{\mathtt{ecc}(x)\}.$$

The central vertices are those vertices whose eccentricity equals the radius. The center of a graph G is the subgraph $Z(G)$ induced on the set of central vertices.

2. The diameter is the maximum eccentricity

$$\mathtt{diam}(G) = \max_{x \in V(G)} \{\mathtt{ecc}(x)\}.$$

The peripheral vertices are those vertices whose eccentricity equals the diameter. The periphery of graph G is the subgraph $\mathtt{per}(G)$ induced on the set of peripheral vertices.

In this application, $\mathtt{ecc}(v) = \{3, 4, 3, 3, 3\}$, $\mathtt{rad}(SC) = 3$, $\mathtt{diam}(SC) = 4$. The center is $Z(G) = \{\dot{l}, \dot{h}, \Delta a_b, a_b\}$. The peripheral vertex is $\dot{p}_c$. The central vertices are shown with big gray points in Fig. 10(a).

Definition 9 (*Strongly Connected Graph*). A directed graph is strongly connected if there is an oriented path between every pair of vertices. A directed graph is weakly connected if there a path between each pair of vertices in the underlying undirected graph.

The articulation vertices of the whole graph whose deletion disconnects the graph are shown in light gray squares. The articulation vertices are $\{\dot{l}, \Delta a_b, a_l\}$.

5.3. *Rooted embedding and graph traversals*

Rooted embeddings are used to represent hierarchies. One vertex is chosen as the root. The other vertices are ranked according to their distance (or depth) to the root. The rooted embedding of the graph is generally drawn choosing a center as a root to have a more balanced embedding (see Fig. 11).

The graph traversals are essential to explore all the vertices and edges of the G and deduce various graph properties. Two different approaches lead to linear time algorithms: the depth-first search (DFS) and the breadth-first search (BFS). The recursive function DFS starts with a vertex (as with a center) and scans its neighbors until the first unexplored vertex is found. The recursive function BFS starts with a vertex and explores all the adjacent vertices to the current vertex and then continues. These explorations on the graph of the CS sub-model, with the central vertex $\dot{h}$ as a root are shown in Fig. 12.

5.4. *Dynamic steady-state graph*

The introduction of the delayed variables is done without making time explicit. Here, we will consider a system close to a steady-state situation where all the variables grow at the same rate. A pseudograph is obtained with self-loops and multiple edges (Fig. 13). A self-loop occurs when a variable is depending on itself in the case of one hysteresis effect. Multiple edges appear when a variable observed at time t on another multiple delayed variables. Moreover the size of the longest circuit is increased to 5 vertices, and we have only one central vertex with a weaker radius $\mathtt{rad}(SC) = 2$. Since $\mathtt{ecc}(v) = \{3, 4, 3, 3, 2\}$, we have $\mathtt{diam}(SC) = 4$. The central vertex is $\{a_b\}$ and the peripheral vertex $\{\dot{p}_c\}$.

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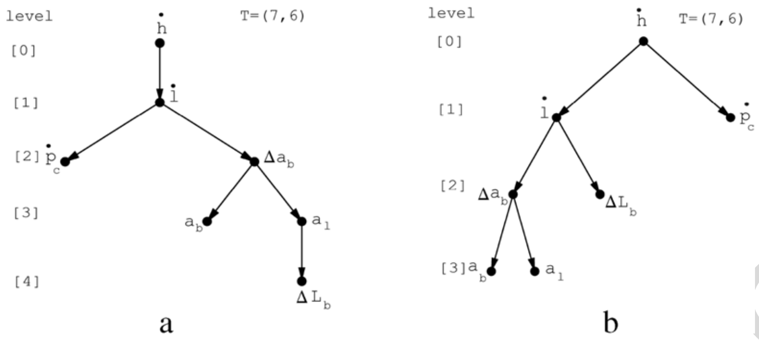

Fig. 12. Depth- and breadth-first search.

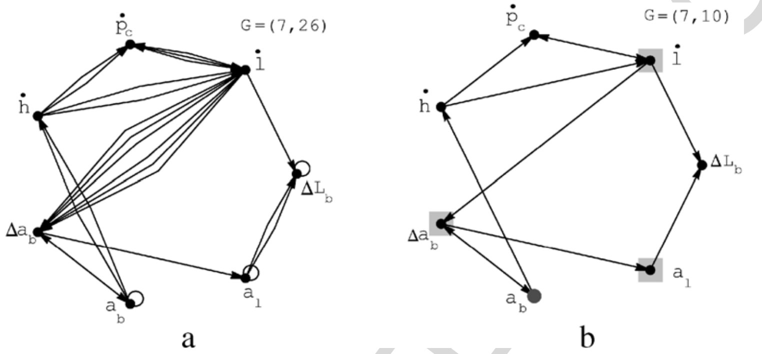

Fig. 13. Pseudograph of the dynamic graph.

Table 1
Typology of variables in the SCs of the static and dynamic sub-model CS.

A	$\dot{h}$ **B***	**B**	**A**	**B***	$\dot{h}$ **B**
$\dot{p}_c$ **D***	$\dot{l}\Delta a_b, a_b$ **C***	**C**	**D***	**C***	a_b **C**
D	**C**	0 variable	$\dot{p}_c$ **D**	a_b **C**	$\dot{l}$ 1 variable

6. Typology based on the graph invariants

6.1. *A typology of vertices*

The typology is based on the all-pairs shortest paths matrix, corresponding to the largest *SC*. The maximum distance of each row is placed in a column vector to the right. This vector expresses the out-eccentricity from which we can deduce the radius and the diameter. The maximum distance of each column is placed in a row vector below. This vector expresses the in-eccentricity from which we deduce the opposite in-radius and the in-diameter. Both criteria are crossed in a 3 × 3 table, where the columns state successively for the periphery *P* of the SC, the center *C* and other points $\bar{P} \bigcup \bar{C}$ with intermediate properties. The rows state for the same in-values with the opposites $\neg P$, $\neg C$, $\neg\bar{P} \bigcup \neg\bar{C}$. We have deduced four types of variables. The type **A** shows an eccentric variable which is weak perturbed and exerts low influences. The type **B*** corresponds to low perturbations but strong influences. The type **C*** states a strong integrated variable which is close to all other interdependent variables. The type **D*** shows a strong dominated variable with a strong perturbation and weak influence.

6.2. *Properties of the CS sub-model*

The results for the sub-model CS are shown in Table 1. In the static to the left, the employment and wages $\{\Delta a_b, a_b, \dot{l}\}$ are integrated variables. The productivity $\dot{h}$ dominates and price $\dot{p}_c$ is a dominated variable. In the dynamic version, the perturbations and influences of the variables are weaker with a similar role.

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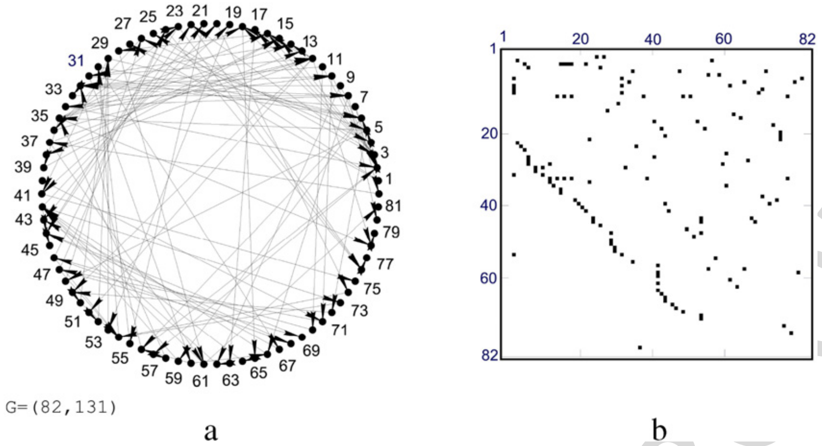

Fig. 14. Initial graph and 82 × 82 adjacency matrix.

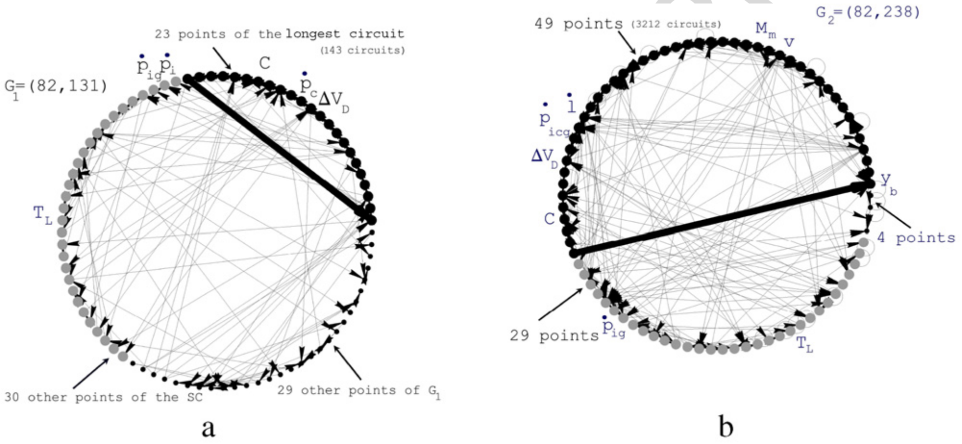

Fig. 15. Graphs of the complete static and dynamic CS model.

7. Extended analysis to large-size models

7.1. Presentation

The complete model for Netherlands (see [1]) consists in 82 equations. This study shows the results for the static and the dynamic steady-state version. The initial circular embedding is shown in Fig. 14(a) for the static version of the model. The numbers and symbols refer to the described variables in the Appendix. The adjacency matrix in Fig. 14(b) shows a sparse graph with a low density of less than 2 per cent.[10]

Improved representations are shown in Fig. 15. The circular embedding is drawn with one of the longest circuits: big black points are those of the chosen longest circuit, big gray points are the remaining vertices of the SC, small black points are the remaining vertices of the complete graph.

7.2. Enumeration of the circuits

The static version has 143 circuits with a maximal length of 23 vertices. The dynamic version has 3212 circuits with a maximal length of 46 vertices. The distribution of the circuits by length in both versions is described in Fig. 16. A longest circuit of the static model is described by the list

$$\left\{y_b, q, \dot{a}_b, \Delta a_b, \Delta w, w, \dot{l}, \dot{p}'_c, \dot{p}_c, \Delta T_K, \Delta N, \Delta V_D, \Delta T_K, T_K, Z, Z^D, C, c, v, \Delta v, n, \Delta n, m_m, y_b\right\}.$$

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[10] This property justifies to prefer an adjacency list representation. The 82-vertex graph consists in 82 lists, with one list for each vertex. Each list records the vertices that are adjacent to the given vertex.

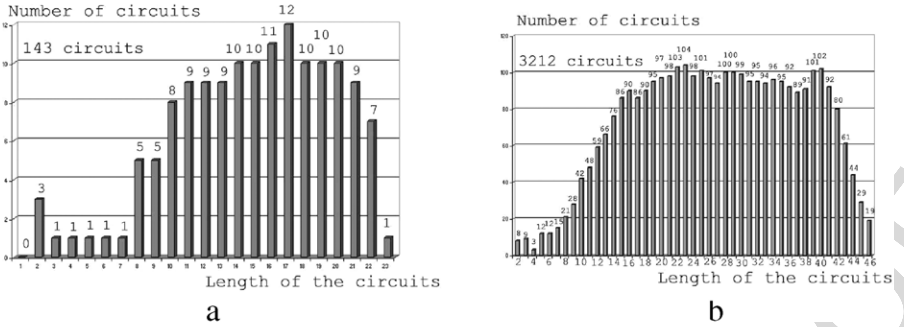

Fig. 16. All circuits of the static(a) and dynamic(b) version.

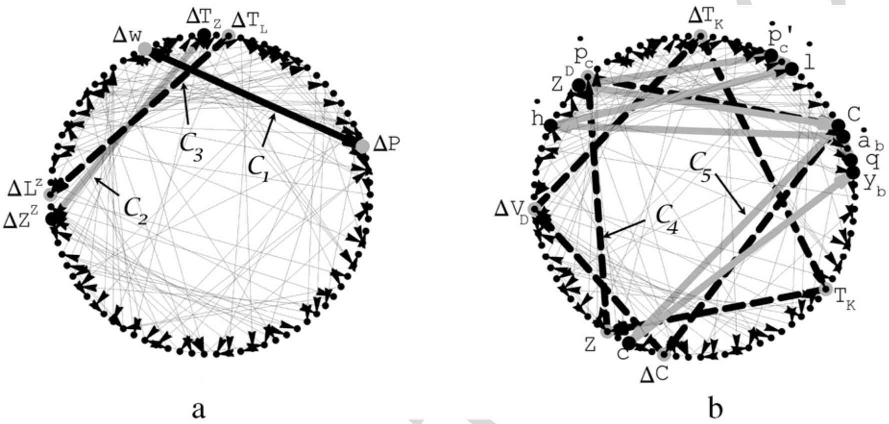

Fig. 17. Short and long edge-disjoint cycles of the static version.

This longest circuit is interpreted as the circular flows of goods and incomes in an open economy with the sequence: production–labor force–wage and price–incomes and taxes–total demand. The dynamic version has 3212 circuits with a maximal length of 46 vertices. A longest circuit of the dynamic model is described by the list

$$\{y_b, y, \dot y, \dot h, \dot K, K, b_m, B_m, B, \Delta B, \Delta Y, r, i_g, v, m_m, M_m, M, \Delta L_q(ex),$$
$$\Delta L_q(g), \Delta L_q, L_q, i, \Delta i, \dot i, \Delta I, I, \dot p_i, \dot a_b, a_b, \Delta w, w, \dot l, \dot p_{i_{cg}}, p_{i_{cg}}, \Delta I_{cg}, \Delta V_D, \Delta T_K, T_K, Z, Z^D, C, \dot p_c, p_c, c, y_b\}.$$

The longest circuit also corresponds to an extended circular flow where the liquidities and investments interact. We have the sequence: supply side (production, productivity, competitiveness)–imports –liquidities–investments–prices–labor market–prices and wages–taxes–revenues–consumption. At this stage, we can verify that the static (or short term) version corresponds to a demand-driven model, and that the dynamic steady-state version shows a supply-driven model.

7.3. Non-edge-disjoint circuits

The static version of the CS model is composed of a maximal list of 5 non-edge-disjoint circuits in Fig. 17(a): the labor market $\mathcal{C}_1 = \{\Delta w, \Delta P, \Delta w\}$, the disposable wage income $\mathcal{C}_2 = \{\Delta L^Z, \Delta T_L, \Delta L^Z\}$, the disposable non wage income $\mathcal{C}_3 = \{\Delta Z^Z, \Delta T_Z, \Delta Z^Z\}$, the circular flow of goods and incomes $\mathcal{C}_4 = \{\Delta C, \Delta V_D, \Delta T_K, T_K, Z, Z^D, C, \Delta C\}$, and the circuit prices –wage – productivity – employment $\mathcal{C}_5 = \{c, y_b, q, \dot a_b, \dot h, \dot l, p'_c, \dot p_c, C, c\}$. The circuits $\mathcal{C}_4$ and $\mathcal{C}_5$ are connected by the consumption C. The circuit $\mathcal{C}_5$ is close to the longest circuit of the CS sub-model.

The dynamic version is composed of 13 non-edge-disjoint circuits in Fig. 17(b). The circuits $\mathcal{C}_1$ to $\mathcal{C}_4$ are the same as in the static version. The circuit $\mathcal{C}_5$ is comparable. The other circuits are: the cycle investment-demand $\mathcal{C}_6 = \{i, v, i\}$ and $\mathcal{C}_7 = \{I, \Delta I, I\}$, the taxes $\mathcal{C}_8 = \{T_L, \Delta T_L, T_L\}$, the cycle of productivity $\mathcal{C}_9 = \{\dot l, \dot a_b, \dot h, \dot l\}$, the cycle money-production $\mathcal{C}_{10} = \{Y, \Delta L_q(g), \Delta Y, Y\}$, the cycle investment-capacities $\mathcal{C}_{11} = \{i, \Delta c_a, c_a, q, i\}$, the cycle of exports $\mathcal{C}_{12} = \{K, b_m, v, n, \Delta n, m_m, M_m, \dot p_c, \dot l, \dot K, K\}$, the cycle of supply $\mathcal{C}_{13} = \{i, y_b, y, \dot y, \dot h, \dot{l}p'_i, \dot p_i, \Delta I, \Delta V_D, \Delta Y, \Delta L_q(ba), \Delta L_q, L_q, i\}$.

7.4. Directed acyclic graphs

The DAGs are shown for the two versions of the CS model in Fig. 18. The vertex Φ in gray is the contraction of the largest SC. In the static version, the variables that dominate the SC are exports (B, b_m), import price (p_m), liquidities and

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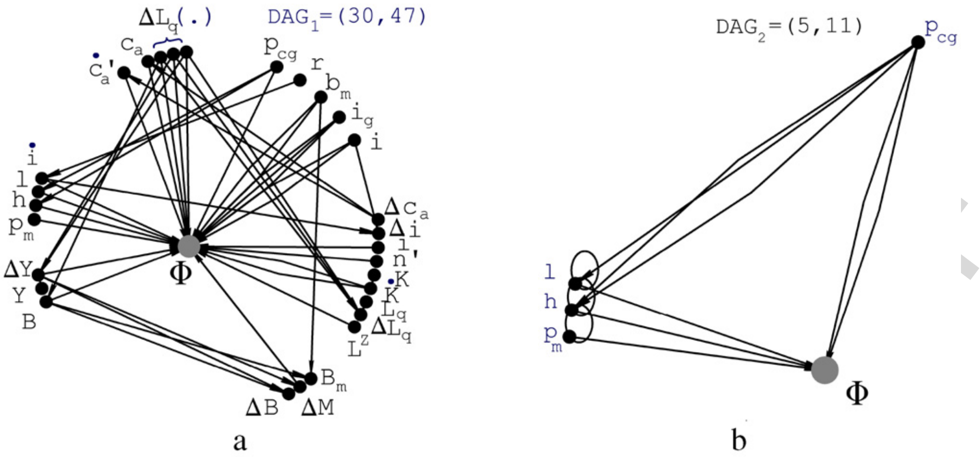

Fig. 18. DAGs of the complete static and dynamic model.

Table 2
The typology of variables in the SCs of the static and dynamic model.

A	**B***	T_L **B**	**A**	**B***	T_L **B**
D*	**C***	$C, \Delta V_D$ **C**	**D***	**C***	y_b, C, v, V_D **C**
$\dot{p}'_i, \dot{p}_{i_g}, \dot{p}_{i_{cg}}$ **D**	$\dot{p}_c$ **C**	47 variables	$\dot{p}_{i_g}, \dot{p}_{i_{cg}}$ **D**	$\dot{l}, M_m$ **C**	69 variables

interest rates $(\Delta L_q, r)$, investments (i, i_g), and supply side (c_a, h). These elements correspond to a short term structure. In the dynamic version, the SC is composed of a large set of 78 vertices. The four remaining vertices of the supply side $\{p_m, p_{cg}, l, h\}$ dominate the SC.

The articulation vertices of the whole two graphs have a common set of vertices with $\{c, c_a, L^D, \dot{h}, l, \Delta L^Z, v, \dot{p}_c, Z^D, \dot{l}, \dot{a}_b\}$. The specific articulation vertices of the static version are $\{\Delta c_a, q, n, \Delta L_q(ba), \Delta L_q(g), \Delta L_q(ex), \Delta Y, M, \Delta I \Delta L_q, \dot{K}\}$. The particular articulation vertices of the dynamic version are $\{i, \Delta V_D\}$.

7.5. Typology of the vertices

Table 2. gives the properties of the SCs which collect the interdependent variables of the model for both versions. In these versions the taxes T_L are weakly dominant, the investment prices $\{p_{i_g}, p_{i_{cg}}\}$ are weakly dominated, the consumption and sales $\{C, \Delta V_D\}$ are weakly integrated. The price $\dot{p}_c$ is a particular weakly integrated vertex of the static version. This variable is the central vertex in the complete CS model, though it was a peripheral in the sub-model.The supply variables $\{y_b, v, M_m, \dot{l}\}$ are particular weakly integrated vertices of the dynamic version.

8. Conclusion

This study provides a useful insight into the structure of macroeconomic models. This knowledge is essential for analysis and economic policy purposes, when looking for instance at the qualitative propagation of some policy measures. The concepts and algorithms of the graph theory are adequate for such an attempt.

Three major but simple proposals are made in this paper: first, a performed graph embedding based on one of the longest circuits, secondly the determination of the maximal list of non-edge-disjoint circuits, and thirdly a typology of vertices which combines the importance of the influences and perturbations to and from the remaining vertices of the graph.

Appendix. Supplement with the CS model

Tables A.1–A.3 show the symbol, the number and definition of each endogenous variable of the CS model. The exogenous variables are not considered in this study. The number of a variable refers to the equation that calculates this variable. The variables refer to levels, unless otherwise indicated. Capital symbols refer to values in current prices, small symbols

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Table A.1

Symbol	Number	Definition
a_b	72	Employment in industries
Δa_b	48	*do*, in absolute variation
$\dot{a}_b$	5	*do*, in percentage change
B	46	Total exports
ΔB	68	*do*, in absolute variation
B_m	70	Merchandise exports
b_m	11	*do*, in volume
C	6	Private consumption
c	56	*do*, in volume
ΔC	59	*do*, in absolute variation
c_a	25	Productive capacity
Δc_a	1	*do*, in absolute variation
$\dot{c}'_a$	27	Productive capacity, adjusted for weather influences
h	39	Labor productivity in industries
$\dot{h}$	35	*do*, in percentage change
I	81	Gross industrial investment in fixed assets, excl. housing
ΔI	61	*do*, in absolute variation
ΔI_{cg}	64	Gross industrial investment by central government
ΔI_g	62	Gross industrial investment by local authorities, incl. housing
i	7	Gross industrial investment in fixed assets, excluding housing
Δi	82	*do*, in absolute variation
$\dot{i}$	37	*do*, in percentage change
i_g	9	Gross industrial investment by local authorities, incl. housing
K	79	Advantage in labor costs over foreign competitors
$\dot{K}$	78	*do*, in percentage change
l	38	Wages level in industries
$\dot{l}$	12	*do*, in percentage change
L^D	29	Total disposable income

Table A.2

Symbol	Number	Definition
L_b	50	Disposable income in industries
$\dot{L}_b$	49	*do*, in percentage change
L_g	52	Public disposable appointments
$\dot{L}_g$	51	*do*, in percentage change
L_q	77	Total amount of liquidities in circulation
ΔL_q	76	*do*, in absolute variation
$\Delta L_{q(ba)}$	22	Creation of liquidities by banks
$\Delta L_{q(g)}$	23	Creation of liquidities by the government
$\Delta L_{q(ex)}$	24	Creation of liquidities via external payments
L^Z	75	Taxable wage income
ΔL^Z	41	Taxable wage income, in absolute variation
M	47	Total imports
ΔM	69	*do*, in absolute variation
M_m	71	Merchandise imports
m_m	10	*do*, in volume
ΔN	66	Goods in stocks in absolute variation
n	8	*do*, in volume
Δn	34	*do*, in absolute variation
n'	80	Goods in stock, excl. live stock
ΔP	4	Labor supply, in absolute variation
p_c	60	Price of private consumption
$\dot{p}_c$	31	*do*, in percentage change
$\dot{p}'_c$	14	*do*, excl. consumption goods imported
p_{cg}	15	Price of government consumption
$\dot{p}_i$	28	Price of industrial investment
$\dot{p}'_i$	16	*do*, excl. investment goods imported
p_{icg}	65	Price level of central government investment
$\dot{p}_{icg}$	18	*do*, in percentage change

to volumes in constant prices. Variables with a dot refer to percentage changes. The symbol Δ denotes an absolute variation.

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Table A.3

Symbol	Number	Definition
p_{ig}	63	Price of local authorities investment, incl. housing
$\dot{p}_{ig}$	17	*do*, in percentage change
p_m	40	Price of total imports
q	3	Surplus capacities
r	13	Rate of long term interest
T_L	53	Taxes on wage income
ΔT_L	19	*do*, in absolute variation
T_K	74	Indirect taxes minus subsidies
ΔT_K	20	*do*, in absolute variation
T_Z	55	Taxes on non-wage income
ΔT_Z	21	*do*, in absolute variation
v	32	Total sales, excluding invisible exports and stock formation
Δv	33	*do*, in absolute variation
V_D	73	Total sales, excl. exports of goods, incl. stock formation
ΔV_D	42	*do*, in absolute variation
w	36	Unemployment
Δw	26	*do*, in absolute variation
Y	45	Gross national product, at market prices
ΔY	44	*do*, in absolute variation
y	57	Gross national product, at market prices
$\dot{y}$	58	*do*, in absolute variation
y_b	2	Gross national product in industries, at market prices
Z	54	Total taxable income
Z^D	30	Disposable non-wage income
ΔZ^D	67	*do*, in absolute variation
ΔZ^Z	43	Taxable non-wage income, in absolute variation

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