Optimisation globale hiérarchique à deux échelons: Application à la détermination des équilibres de Nash-Stackelberg

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1. Decision-making in hierarchical organizations
2. Decision-making problem
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INTRODUCTION

1. Decision-making in hierarchical organizations: the decision process

- Suppose an organization with 2 levels of hierarchy in decision making: the **leader** at the upper-level and the **follower** at the lower-level.
- Example: a private company, in which the **top management** has overall economic objectives and where the **spezialized divisions** have productivity and marketing objectives.

1. Decision-making in hierarchical organization: the equivalent game

- The decision process is such that the two levels proceed sequentially: the **leader** may influence the follower's decision, and the **follower** observes the leader's decision and reacts optimally.
- This problem is similar to the static noncooperative 2-person game by Stackelberg. The 2 players optimize their own payoff function by controlling their **decision variables**. Both players have a **perfect information** about the objectives and strategies of the opponent. The **leader plays first**, but must anticipate all the possible reactions of his opponent and the **follower reacts** optimally.

2. Decision-making problem

The formulation of this decision-making problem refers to a bilevel programming (BLP) problem, in which the constraint region is implicitly determined by another optimization problem. Let the decision variables controlled by the leader be $\mathbf{x} \in X \subseteq \mathbb{R}^n$ and the follower's decision variables be $\mathbf{y} \in Y \subseteq \mathbb{R}^m$. A general form of the BLP may be written with no upper-level constraints, such as

$$\left[\begin{array}{l} \min\limits_{\mathbf{x}\in X} \text{imize} \;\; F(\mathbf{x},\mathbf{y}) \equiv \mathbf{c}_1^T \mathbf{x} + \mathbf{d}_1^T \mathbf{y} \\ \quad s.t. \left[\begin{array}{l} \min\limits_{\mathbf{y}\in Y} \text{imize} \;\; f(\mathbf{x},\mathbf{y}) \equiv \mathbf{c}_2^T \mathbf{x} + \mathbf{d}_2^T \mathbf{y} \\ s.t. \;\; \mathbf{g}(\mathbf{x},\mathbf{y}) \equiv \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} - \mathbf{b} \le 0 \end{array}\right. \end{array}\right.$$

where the outer and inner objective functions are $F, f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ and the inner constraints $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^q$. All the functions are assumed to be continuous and C^2. Moreover, the sets X and Y will place additional restrictions on the variables such as bounds, nonnegativity, or integrality.

Under convexity and regularity conditions, using the Karush-Kuhn-Tucker (KKT) conditions for the second level problem, the BLP is reformulated as a single nonlinear optimization problem

$$\left.\begin{array}{r} \min\limits_{\mathbf{x},\mathbf{y},\mathbf{u}} \text{imize} \;\; \mathbf{c}_1^T \mathbf{x} + \mathbf{d}_1^T \mathbf{y} \\ s.t. \;\; \nabla_{\mathbf{y}} \mathcal{L}(\mathbf{x},\mathbf{y},\mathbf{u}) = \mathbf{d}_2^T + \mathbf{u}^T \mathbf{B} = 0 \\ \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} - \mathbf{b} \le \mathbf{0} \\ u_i \left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} - \mathbf{b}\right)_i = 0, i = 1, \ldots, q \\ \mathbf{x} \ge \mathbf{0}, \mathbf{y} \ge \mathbf{0}, u_i \ge 0, i = 1, \ldots, q \end{array}\right]$$

where

$$\mathcal{L}(\mathbf{x},\mathbf{y},\mathbf{u}) = f(\mathbf{x},\mathbf{y}) + \sum_{i=1}^{q} u_i g_i(\mathbf{x},\mathbf{y})$$

denotes the Lagrangean associated with the lower-level problem and $\mathbf{u}$ the multipliers.

3. Bard's BLP example

The BLP problem is

$$\begin{cases} \min\limits_{x\geq 0}\text{imize} \;\; F(x,y) \equiv x - 4y \\ s.t. \; \begin{cases} \min\limits_{y\geq 0}\text{imize} \;\; f(y) \equiv y \\ s.t. \;\; g_1(x,y) \equiv -x - y + 3 \leq 0 \\ \quad\;\; g_2(x,y) \equiv -2x + y \leq 0 \\ \quad\;\; g_3(x,y) \equiv 2x + y - 12 \leq 0 \\ \quad\;\; g_4(x,y) \equiv -3x + 2y + 4 \leq 0 \end{cases} \end{cases}$$

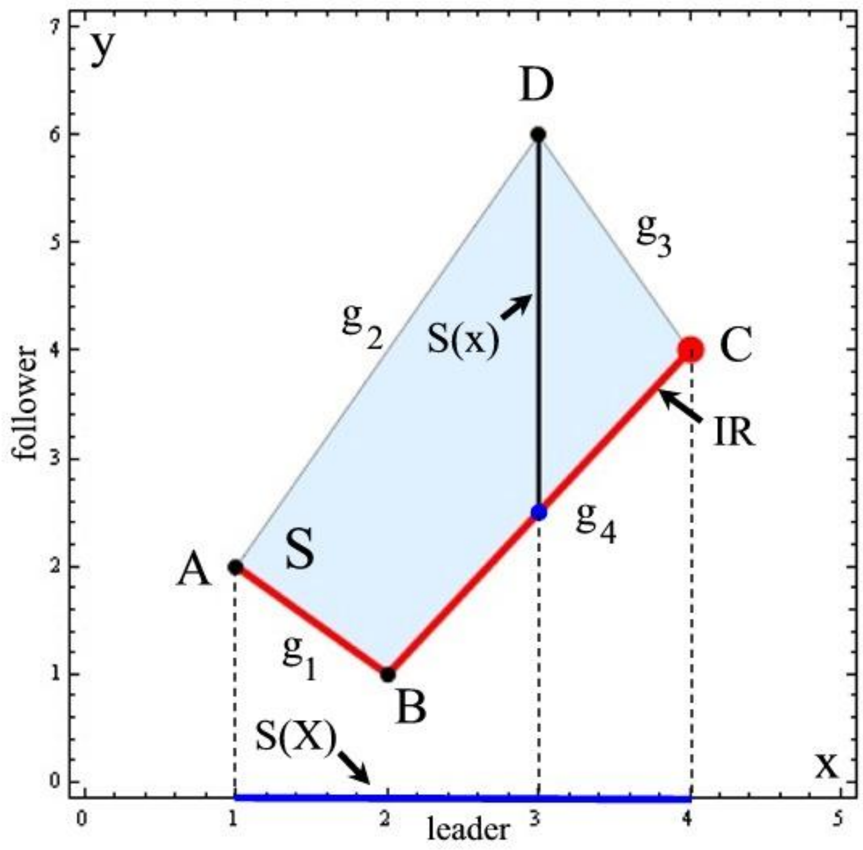
The polyhedron S is the set

$$S = \{(x,y) : x \in X, y \in Y, g_i(x,y) \leq 0, i \in \mathbb{N}_4\}.$$

For fixed $x \in X$, the follower's feasible region is the set

$$S(x) = \{y \in Y : y \geq 3 - x, y \leq 2x, y \geq -12 + 2x, y \leq 2 - 1.5x\}.$$

The follower's reaction set is defined by the set $RR(x) = \{y \in \arg\min f(x,\hat{y}) : \hat{y} \in S(x)\}$.The piecewise inductible region $IR = \{(x,y) : x \geq 0, y \in RR(x)\}$ is the feasible set of the leader. The projection of S onto the leader's decision space is the set $S(X) = \{x \in X : \exists y \in Y, \mathbf{g}(x,y) \leq 0\}$. We deduce $S(X) = \{x \in [1,4]\}$.



3. Bard's BLP example

Theorem 1. The inductible region IR is equivalently a piecewise linear equality constraint comprised of supporting hyperplanes of S. **Proof.** see Bard [5], p. 199.■

Theorem 2. The solution $(\mathbf{x}^*, \mathbf{y}^*)$ of a linear BLP occurs at a vertex of S. **Proof.** see Bard [5], p. 200. ■

The payoff matrix in TABLE 1 is used to search for the optimum. For feasible points, the payoffs are (f, F) and ∞ for non feasible points. For extreme points on the IR, the payoffs are highlighted.

The optima occur at extreme points of inductible region. The global optimal solution is achieved at C (4,4) with $(f^*, F^*) = (4, -12)$ .There is no Pareto optimal. A local optimum is at A(1,2).

TABLE 1 PAYOFF MATRIX OF EXAMPLE 1

		Leader (x)				min
		1	2	4	6	
Follower (y)	1	∞	(2,-7)	∞	∞	2
	2	(1,-2)	(2,-6)	(4,-14)	∞	1^*
	3	∞	∞	(4,-13)	(6,-21)	4
	4	∞	∞	(4,-12)	∞	4^+
min		-2	-7	-14^+	-21^*	

CONTENTS

1. Hierarchical programming problem

1.1 *Multi-level programming*

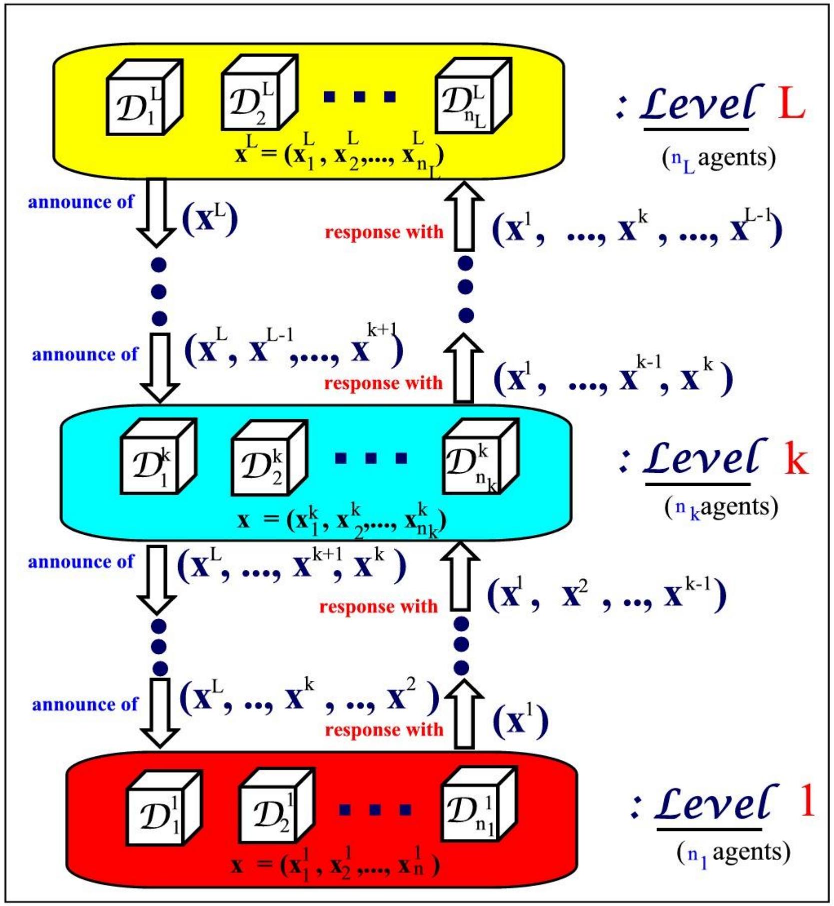
The structure of an N-**agent L-level programming problem** is shown in Fig. 2. The hierarchical decision system consists of L levels. At each level $k = 1,\dots,L$, the number of agents (or divisions) is n_k. Agent i at level k is denoted by $\mathcal{D}_i^k \in \mathcal{L}^k$ and all other agents at the same level will be $\mathcal{D}_{-i}^k$. Agent $\mathcal{D}_i^k$ is controlling $\mathbf{x}_i^k \in X_i^k \subseteq \mathbb{R}^{m_{k_i}}$ with m_{k_i} decision variables. The decision variables for level k are

$$\mathbf{x}^k = \left(\mathbf{x}_1^k,\dots,\mathbf{x}_{n_k}^k\right) \in X^k = \times_{i=1}^{n_k} X_i^k .$$

The total number of decision variables in the system is

$$N = \sum_{k=1}^{L} \sum_{i=1}^{n_k} m_{k_i} .$$

The system is **a nested collection of Nash equilibrium problems**: within each level, agents play an $n-$person nonzero-sum game and between levels, the decision process is similar to an $n-$person Stackelberg game. Every agent has a **perfect information**. They are perfectly informed about the decisions at upper levels, but not at their levels or below. The agents of one level will also influence the other agents at lower levels, via their objective functions and the sets of feasible decisions.



1. Hierarchical programming problem

1.2 *Nash-Stackelberg equilibrium solution*

Let the objective functions of agent $\mathcal{D}_i^k$ be $f_i^k : \mathbb{R}^N \mapsto \mathbb{R}$ for $i = 1,\ldots,n_k$ and $k = 1,\ldots,L$.

Definition 1 (Nash equilibrium responses). *The Nash equilibrium responses at level k of $f_1^k,\ldots,f_{n_k}^k$ over the compact set S, for each $k = 1,\ldots,L-1$, is defined as.*

$$\Psi^k(S) \equiv \left\{ \hat{w} \in S \;\middle|\; \begin{array}{l} f_1^k(\hat{w}) = \underset{x_1^k, \hat{x}_{-1}^k : \hat{x}^L \ldots \hat{x}^{k+1}}{\text{maximize}} f_1^k(w) \\ \vdots \\ f_{n_k}^k(\hat{w}) = \underset{x_1^k, \hat{x}_{-1}^k : \hat{x}^L \ldots \hat{x}^{k+1}}{\text{maximize}} f_{n_k}^k(w) \end{array} \right\}.$$

Assumption 1. *The parametric problem for the n_k agents $\mathcal{D}_i^k \in \mathcal{L}^k$ has no multiple equilibrium solution. That is, for fixed values $\left(\hat{\mathbf{x}}^L, \hat{\mathbf{x}}^{L-1},\ldots,\hat{\mathbf{x}}^{k+1}\right)$, the set $\Psi^k(S)$ has at most one element .*

Definition 2 (Equilibrium solution). *Let the level k feasible set be $S^k \equiv \Psi^{k-1}\left(S^{k-1}\right)$ for any given $\left(\hat{\mathbf{x}}^L, \hat{\mathbf{x}}^{L-1},\ldots,\hat{\mathbf{x}}^{k+1}\right)$. An equilibrium solution $\mathbf{x} \in S^k$ is*

$$\mathbf{x} \equiv \left(\mathbf{x}^1,\ldots,\mathbf{x}^{k-1},\mathbf{x}_i^k,\mathbf{x}_{-i}^k,\mathbf{x}^{k+1},\ldots,\mathbf{x}^L\right)$$

Definition 3 (Stackelberg feasibility). If the equilibrium solution x also satisfies the rational responses of lower levels $\mathcal{L}^{k-1},\ldots,\mathcal{L}^1$, then it is Stackelberg feasible.

The programming problem to be solved simultaneously by every $\mathcal{D}_i^k \in \mathcal{L}^k$ to get a Nash-Stackelberg solution is

$$\left.\begin{array}{ll} \underset{x_i^k, x_{-i}^k : \hat{x}^L, \hat{x}^{L-1},\ldots,\hat{x}^{k+1}}{\text{maximize}} & f_i^k(\mathbf{x}),\ i = 1,\ldots,n_k \\ & s.t.\ \mathbf{x} \in S^k \subset \mathbb{R}^N \end{array}\right]$$

$$\mathcal{P}^L \left[\begin{array}{l} \underset{\mathbf{x}_i^L,\mathbf{x}_{-i}^L}{\max\text{imize}}\ f_i^L(\mathbf{x}),\ i = 1,\ldots,n_L \\ \quad where\ \mathbf{x}^{L-1},\ldots,\mathbf{x}^1\ solve: \\ \mathcal{P}^{L-1}\left[\begin{array}{l} \underset{\mathbf{x}_i^{L-1},\mathbf{x}_{-i}^{L-1}:\mathbf{x}^L}{\max\text{imize}}\ f_i^{L-1}(\mathbf{x}),\ i = 1,\ldots,n_{L-1} \\ \quad where\ \mathbf{x}^{L-2},\ldots,\mathbf{x}^1\ solve: \\ \quad\quad \ddots \\ \mathcal{P}^2\left[\begin{array}{l} \underset{\mathbf{x}_i^2,\mathbf{x}_{-i}^2:\mathbf{x}^L,\ldots,\mathbf{x}^3}{\max\text{imize}}\ f_i^2(\mathbf{x}),\ i = 1,\ldots,n_2 \\ \quad where\ \mathbf{x}^1\ solves: \\ \mathcal{P}^1\left[\begin{array}{l} \underset{\mathbf{x}_i^1,\mathbf{x}_{-i}^1:\mathbf{x}^L,\ldots,\mathbf{x}^2}{\max\text{imize}}\ f_i^1(\mathbf{x}),\ i = 1,\ldots,n_1 \\ \quad s.t.\ \mathbf{x} \in S \end{array}\right. \end{array}\right. \end{array}\right. \end{array}\right.$$

1. Hierarchical programming problem

1.3 *Resolution methods*

Several algorithms have been proposed in the literature for solving linear and nonlinear BLPs. These algorithms are relevant to **three main approaches**: the extreme vertex exploration in the linear case, the reformulation of the original problem, the descent methods.

- **Extreme-point approach**. According to **Theorem 2**, some form of vertex enumeration may be employed for linear BLPs. The algorithms are based on the **vertex enumeration and evaluation of extreme points** of the constraint region. The *K*th best method (Bialas and Karwan [8], Candler and Townsley [11]) considers bases of the relaxed problem (complementarity term omitted), sorted in increasing order of the upper level objective function values.
- **Reformulation techniques**. Suppose that **the lower-level problem is convex and regular**. Then, the original problem is transformed into a single optimization problem as in section I.B, by employing the **KKT conditions** of the lower-level problem (Lu et al. [22, 23]. However, the nonconvexities, that occur in the linear complementarity slackness constraint, require some further transformations, such as: adding new variables and constraints and solving a mixed integer programming technique (Fortuny-Amat and McCarl [19]) or using a B&B enumeration technique (Bard and Falk [4]). Also, by using the KKT conditions for the lower-level problem, the parametric complementary pivot (PCP) has been proposed: in Bialas et al. [9] it is updating a parameter α, which bounds the upper-level objective function value. The lower-level problem may also be replaced by a penalized problem. Penalty methods in Aiyoshi and Shimizu [1,2]. White and Amandalingam [30] introduce the duality gap of the follower's problem into the leader's problem.
- **Descent methods**. Assume that the optimal solution at lower-level is unique, and define an implicit function **y** (the decision variables of the leader) of **x** (the decision variables of the follower). Given a feasible point, an attempt is made to find a feasible direction along which the upper-level objective decreases. The main issues is the availability of the gradient at the feasible point (Falk and Liu [17], Shi et al. [25, 26], Vicente et al. [27].).

Method:

***K*th-Best algorithm** is for computation of the global solution of a BLP problem with one follower, by enumerating the extreme points of the constraint region. Let the N ordered basic solutions to the LP problem

$\underset{x,y}{\text{minimize}}\ c_1x + d_1y : (x,y) \in S$ such that

$c_1x_{[i]} + d_1y_{[i]} \le c_1x_{[i+1]} + d_1y_{[i+1]}, i = 1,\dots,N-1$. Then solving is equivalent to

finding the index $K^* = \min\left\{i \in (1,\dots,N) : \left(x_{[i]}, y_{[i]}\right) \in IR\right\}$.

Algorithm:

- Step 1: **Put** $i \leftarrow 1$. **Solve** the LP problem :

  $\underset{x,y}{\text{minimize}}\left\{c_1x + d_1y : (x,y) \in S\right\}$. **Get** the optimal solution $\left(x_{[i]}, y_{[i]}\right)$

  Let the sets $W = \left\{\left(x_{[i]}, y_{[i]}\right)\right\}$ and $T = \{\varnothing\}$. Go to Step 2.
- Step 2: **Solve** the follower's problem $\underset{y}{\text{minimize}}\left\{d_2y : y \in P\left(x_{[i]}\right)\right\}$

  with $x = x_{[i]}$. **Get** the optimal solution $\bar{y}$. **If** $\bar{y} = y_{[i]}$ **then** STOP and

  $\left(x_{[i]}, y_{[i]}\right)$ is the global optimum to the BLP problem with $K^* = i$.

  Otherwise, Go to Step 3.
- Step 3: **Let** $W_{[i]}$ be the set of adjacent extreme points of $\left(x_{[i]}, y_{[i]}\right)$

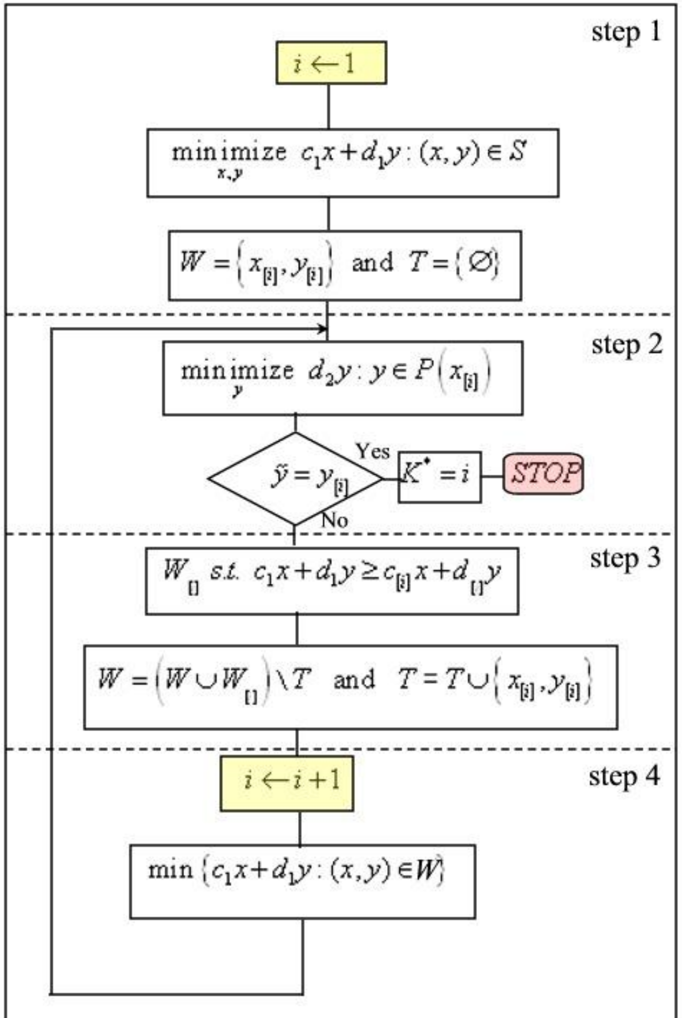
  such that $(x,y) \in W_{[i]} \Rightarrow c_1x + d_1y \ge c_1x_{[i]} + d_1y_{[i]}$. **Let**

  $T = T \cup \left\{\left(x_{[i]}, y_{[i]}\right)\right\}$ and $W = \left(W \cup W_{[i]}\right) \setminus T$. Go to Step 4.
- Step 4. **Set** $i \leftarrow i+1$. **Choose** $\left(x_{[i]}, y_{[i]}\right)$ so that

  $c_1x + d_1y = \min\left\{c_1x + d_1y : (x,y) \in W\right\}$. Go to Step 2.

1. Hierarchical programming problem

1.3 *Resolution methods: Kth Best approach*



Bard's example:

$$\begin{cases} \underset{x\in X}{\text{minimize}}\ F(x,y) = x-4y \\ \begin{cases} s.t.\ \underset{y\in Y}{\text{minimize}}\ f(y)=y \\ g_{21}(x,y)\equiv 3-x-y\le 0 \\ g_{22}(x,y)\equiv -4+3x-2y\le 0 \\ g_{23}(x,y)\equiv -2x+y\le 0 \\ g_{24}(x,y)\equiv -12+2x+y\le 0 \end{cases} \end{cases}$$

Resolution:

Solve the leader's problem $\underset{x,y}{\text{minimize}}\{F(x,y)\equiv x-4y : (x,y)\in S\}$.

Find the solution $(x_{[1]}, y_{[1]}) = (3,6)$. **Place** the solution into the extreme-point set W, we then have the set $W=\{(3,6)\}$. The set T of already examined extreme points is $T=\{\varnothing\}$.

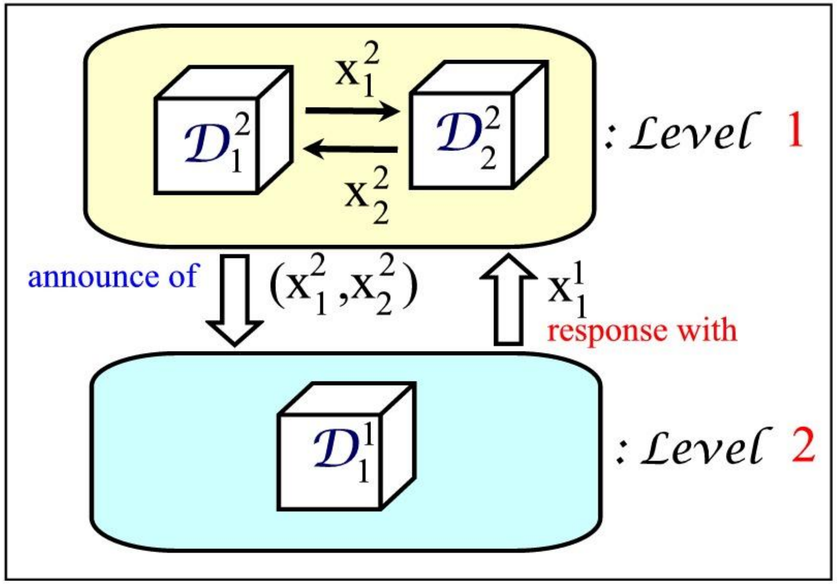
- Loop 1: Let $i\leftarrow 1$. Solve the follower's problem with $x=3$ such as $\underset{y}{\text{minimize}}\{f(y)\equiv y : x=3, (x,y)\in S\}$. Get the solution $\tilde{y}=2.5\ne y_{[1]}(=6)$. Find the adjacent extreme-points and get the set $W_{[1]}=\{(3,6),(4,4),(1,2)\}$. We have also $T=\{(3,6)\}$ and $W=\{(4,4),(1,2)\}$. Choose $(x_{[1]}, y_{[1]})=(4,4)$. Update $i\leftarrow 2$. Then Go to Step 2.
- Loop 2: Solve the follower's problem with $x=4$ $\underset{y}{\text{minimize}}\{f(y)\equiv y : x=4, (x,y)\in S\}$. Get the solution $\tilde{y}=4=y_{[2]}$. Then STOP.

2. Nash-Stackelberg game application:

2.1 *Bialas & Karwan's example*

Example 2. This simple application is taken from Yang and Bialas [31]. The hierarchical system consists of two levels, with **one leader** at level 1 and **two followers** at level 2. The vector of decision variables is $\mathbf{x} = \left(x_1^1, x_1^2, x_2^2\right)$. The leader $\mathcal{D}_1^1$ controls x_1^1 and the followers $\mathcal{D}_1^2$, $\mathcal{D}_2^2$ control x_1^2, x_2^2 respectively. The objective functions of the players $f_1^1, f_1^2, f_2^2 : \mathbb{R}^3 \mapsto \mathbb{R}$ are all linear and the nonnegative decision variables must satisfy a set of five linear constraints $\mathbf{g} : \mathbb{R}^3 \mapsto \mathbb{R}^5$.

$$\mathcal{P}^2 \left[\begin{array}{l} \underset{x_1^2, x_{-1}^2}{\max imize} \;\; f_1^2(\mathbf{x}) \equiv 2x_1^1 + 0.7x_1^2 - 0.6x_2^2 \\ \underset{x_2^2, x_{-2}^2}{\max imize} \;\; f_2^2(\mathbf{x}) \equiv -2x_1^1 + 2x_1^2 - 1.5x_2^2 \\ \qquad\qquad\qquad \textit{where} \;\; x_1^1 \;\; \textit{solves}: \\ \mathcal{P}^1 \left[\begin{array}{l} \underset{x_1^1 : \hat{x}_2^2, \hat{x}_{-2}^2}{\max imize} \;\; f_1^1(\mathbf{x}) \equiv x_1^1 + 0.8x_1^2 + 1.2x_2^2 \\ s.t. \;\; g_1(\mathbf{x}) \equiv -3 + x_1^1 + x_1^2 + x_2^2 \le 0 \\ \qquad g_2(\mathbf{x}) \equiv 1 + x_1^1 - x_1^2 - x_2^2 \le 0 \\ \qquad g_3(\mathbf{x}) \equiv -1 + x_1^1 + x_1^2 - x_2^2 \le 0 \\ \qquad g_4(\mathbf{x}) \equiv -1 + x_1^1 - x_1^2 + x_2^2 \le 0 \\ \qquad g_5(\mathbf{x}) \equiv -0.5 + x_1^1 \le 0 \\ \qquad x_1^1, x_1^2, x_2^2 \ge 0 \end{array} \right. \end{array} \right.$$



2. Nash-Stackelberg game application:

2.2 *Rational reaction sets*

The **best responses** BR_1^2 and BR_2^2 of the two followers are the heavy lines [A-E-F-G-C] and [D-A-B] respectively. This lines are deduced from TABLE 1. The best responses of agent $\mathcal{D}_1^2$ for each possible choice of x_2^2 by $\mathcal{D}_2^2$ such that $x_2^2 \in \{0, 0.5, 1, 1.5, 2\}$ are calculated as follows.

If $\mathcal{D}_2^2$ chooses $x_2^2 = 0$, the response of $\mathcal{D}_1^2$ is $x_1^2 = 1$ (this is the only possibility) and the subsequent response of the leader $\mathcal{D}_1^1$ to these choices $(x_1^2, x_2^2) = (1, 0)$ is $x_1^1 = 0$. **If** $\mathcal{D}_2^2$ chooses $x_2^2 = 1$ (see highlighted numbers in TABLE 2), the response of $\mathcal{D}_1^2$ is multiple with $x_1^2 \in \{0, 0.5, \mathbf{1.5}, 2\}$.

The corresponding payoffs for $\mathcal{D}_1^2$ being $f_1^2 \in \{-0.6, 0.75, \mathbf{1.45}, 0.8\}$, this player will retain the maximum and then chooses $x_1^{2*} = 1.5$. The subsequent response of the leader $\mathcal{D}_1^1$ to these choices $(x_1^2, x_2^2) = (1.5, 1)$ is $x_1^1 = 0.5$. The calculation goes on by using the same rules.

TABLE 1. DATA FOR EXAMPLE 2

Point	Coordinates			Function values		
	x_1^1	x_1^2	x_2^2	f_1^1	f_1^2	f_2^2
A	0	1	0	0.8	0.7	2
B	0	2	1	2.8	0.8	2.5
C	0	1	2	3.2	-0.5	-1
D	0	0	1	1.2	-0.6	-1.5
E	0.5	1	0.5	1.9	1.4	0.25
F	0.5	1.5	1	2.9	1.45	0.5
G	0.5	1	1.5	3.1	0.8	-1.25
H	0.5	0.5	1	2.1	0.75	-1.5

2. Nash-Stackelberg game application:

2.3 *Nash-Stackelberg equilibrium*

The point $A \in \{BR_1^2 \cap BR_2^2\}$. Is a Nash equilibrium. Since it is an element of S^2, it is also a Stackelberg solution. This example is solved by using a 6 steps enumeration procedure in Yang and Bialas [31].