Convex Underestimating Relaxation Techniques for Nonconvex Polynomial Programming Problems

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Abstract— **This paper introduces to constructing problems of convex relaxations for nonconvex polynomial optimization problems. Branch-and-bound algorithms are convex relaxation based. The convex envelopes are of primary importance since they represent the uniformly best convex underestimators for nonconvex polynomials over some region. The reformulation-linearization technique (RLT) generates LP (linear programming) relaxations of a quadratic problem. The LP-RLT yields a lower bound on the global minimum. RLT operates in two steps: a reformulation step and a linearization (or convexification) step. In the reformulation phase, the constraints (constraints and bounds inequalities) are replaced by new numerous pairwise products of the constraints. In the linearization phase, each distinct quadratic term is replaced by a single new RLT variable. This RLT process produces an LP relaxation. LMI formulations (linear matrix inequalities) have been proposed to treat efficiently with nonconvex sets. An LMI is equivalent to a system of polynomial inequalities. A semialgebraic convex set describes the system. The feasible sets are spectrahedra with curved faces, contrary to the LP case with polyhedra. Successive LMI relaxations of increasing size can be used to achieve the global optimum. Nonlinear inequalities are converted to an LMI form using Schur complements. Optimizing a nonconvex polynomial is equivalent to the LP over a convex set. Engineering application interests include system analysis, control theory, combinatorial optimization, statistics, and structural design optimization.**

Keywords—convex relaxation; polynomial optimization; nonconvex optimization; LMI formulation; structural optimization

I. INTRODUCTION

This paper introduces to the problem of constructing convex relaxations for nonconvex polynomial optimization problems. Techniques such as outer-approximation, branch-and-bound (B&B) algorithms, reformulation-convexification methods are convex relaxation based [1].

Convex extensions and envelopes are of primary importance to the efficiency of global optimization methods. These notions reflect the capability to construct tight convex relaxations[1]. Locatelli [3] determines convex envelopes for quadratic and polynomial functions over polytopes. Convex underestimators of nonconvex functions over some region are essential to B&B techniques. However, computing convex envelopes is NP-hard, even for simple polynomials[2]. The nuclear norm (i.e., the sum of singular values) heuristic is also used instead of the convex envelope of the objective function. The affine matrix rank minimizing problem (RMP) uses the nuclear norm of the rank function. In this case, the nonconvex objective rank function is replaced by its convex envelope (i.e., the nuclear norm) [4]. In statistics, this important practical problem may consist of finding the least complex stochastic model, which is consistent with observations and priors.

The reformulation-linearization technique (RLT) generates LP (linear programming) relaxations of a quadratic problem [5]. The LP-RLT yields a lower bound on the global minimum. RLT operates in two steps: a reformulation step and a linearization (or convexification) step. In the reformulation phase, the constraints (constraint and bound inequalities) are replaced by new pairwise products of the constraints (i.e., bound factor product, bound-constraint factor product, and constraint factor product inequalities). In the linearization phase, each distinct quadratic term is replaced by a single new RLT variable. This RLT process produces an LP relaxation.

LMI (linear matrix inequalities) formulations have been proposed to treat efficiently with nonconvex sets. An LMI is equivalent to a system of polynomial inequalities. A semialgebraic convex set describes the system. The feasible sets are spectrahedra with curved faces, contrary to the LP case with polyhedra. SOS (sum of squares) relaxations can be used to obtain good approximate SDP (semidefinite programming) descriptions of convex envelopes (e.g., computing the convex envelope of quadratic forms over polytopes via a semidefinite program). Successive LMI relaxations of increasing size can be used to achieve the global optimum[3]. Nonlinear inequalities are

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[1] The theory of convex extensions is developed for lower semi-continuous functions in [2].

[2] A proposition may consist in computing the convex envelopes over simpler domains such as triangles. Some examples are proposed in [3].

[3] The approach consists in approximating a programming problem (PP) by a sequence of easier relaxed problems, such that the sequence of solutions

converted to an LMI form using Schur complements. Optimizing a nonconvex polynomial is equivalent to the LP over a convex set.

Engineering application interests include system analysis, control theory, combinatorial optimization, statistics and structural design. As a practical illustration, one can mention the truss topology design problem. This problem can be set to an equivalent LMI, by using the Schur lemma for linearization.

This article is organized as follows. Section II introduces to some important convex transforms in practice, such as the eigen-transformation, the convex envelopes, the nuclear norm and the conjugacy transform. The basic reformulation-linearization technique is presented in Section III for nonconvex QP (quadratic programming) problems. An illustrative numerical example is solved in Appendix A. The effectiveness of semidefinite programming (SDP) in polynomial optimization is shown in Section IV. The following essentials aspects are introduced: the LMI feasibility sets, the LMI formulation of SOS (sum of squares) polynomials, and simplified engineering application to this approach. Appendix B is devoted to the SDP interpretation of quadratic optimization problems.

II. CONVEX TRANSFORMS

Convexification transformation methods can convert a nonconvex problem to an equivalent problem, such as a concave minimization problem, a reverse convex minimization problem or a difference convex (d.c.) programming problem. The followings are restricted to concepts such as the eigen-transformation, convex envelopes, nuclear norm and conjugacy transformations.

A. Eigen-Transformation [10]

Let QP problem be

$$\mathbf{QP}: \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}}\ \mathbf{c}^T\mathbf{x}+\mathbf{x}^T\mathbf{Q}\mathbf{x}$$
$$\text{subject to}:$$
$$\mathbf{Ax}\le\mathbf{b},$$
$$x_k\in\left[l_k,u_k\right],\ k=1,\dots,n.$$

The eigen-transformation for the QP problem is a particular linear transformation based on the eigenstructure of the quadratic objective. Let $\mathbf{Q}=\mathbf{PDP}^T$ where $\mathbf{D}$ is diagonal with eigenvalue elements of $\mathbf{Q}$, and $\mathbf{P}$ column eigenvectors. Define $\mathbf{x}=\mathbf{Pz}$, so that $\mathbf{z}=\mathbf{P}^T\mathbf{x}$. The resulting eigen-transformed QP is

$$\underset{\mathbf{z}\in\mathbb{R}^n}{\text{minimize}}\ \mathbf{c}^T\mathbf{Pz}+\mathbf{z}^T\mathbf{Dz}$$
$$\text{subject to}:$$
$$\mathbf{APz}\le\mathbf{b},$$
$$\mathbf{l}\le\mathbf{Pz}\le\mathbf{u}.$$

B. Convex Envelope

Definition 1. The convex envelope for a nonconvex function f and region X is the largest convex underestimator of f over X, so that

$$\text{conv}_{f,X}=\sup\left\{c(\mathbf{x}):c(\mathbf{x}')\le f(\mathbf{x}'),\forall\mathbf{x}'\in X\subset\mathbb{R}^n\right\},$$

where $c(.)$ is a convex function. □

The convex envelope can be a convex polyhedral representation, i.e., the maximum of a finite number of affine underestimators. In [11], Locatelli and Schoen derive convex envelopes of bivariate functions $f(x,y)$ over general two-dimensional polytopes, assuming that some conditions on f are satisfied. Carathéodory's theorem yields the convex envelope of f at a point $K\in P$. Given a polytope $P\subset\mathbb{R}^n$ and a function f, we have the PP

$$\text{conv}_{f,P}(K)=\min\left\{\sum_{i=1}^{n+1}\lambda_i f(Q_i):Q_i\in P,\ i=1,\dots,n+1\right\},$$
$$\text{subject to}:$$
$$\sum_{i=1}^{n+1}\lambda_i=1,\ \sum_{i=1}^{n+1}\lambda_i Q_i=K,\ \lambda_i\ge 0.$$

Theorem 1 [1], pp. 45-46. Let $f(\mathbf{x})$ be a lower semicontinuous function defined on the convex compact set $X\subset\mathbb{R}^n$ and $\phi(\mathbf{x})$ be the convex envelope of f on X, then we have

(i) $\underset{\mathbf{x}\in X}{\text{minimize}}\ f(\mathbf{x})=\underset{\mathbf{x}\in X}{\text{minimize}}\ \phi(\mathbf{x})=\hat{f}$

(ii) $\left\{\mathbf{y}\in X: f(\mathbf{y})=\hat{f}\right\}\subseteq\left\{\mathbf{y}\in X:\phi(\mathbf{y})=\hat{f}\right\}$.□

Hence, the theorem states that for each nonconvex PP on a convex feasible region, one can associate a convex PP for which we have the same optimal solution.

Example 1. Let the nonconvex polynomial of degree 4 in Fig.1

$$f(x)=1.5+2.882x-1.277x^2+0.096x^3+0.005x^4$$

where $x\in[0,7]$. The convex envelope is

$$\text{conv}\,f(x)=\begin{cases}1.5-0.489, \text{ if } x\in[0,4.83]\\ f(x), \text{ if } x\in[4.83,7].\end{cases}$$

converge to a global solution of the global optimization problem. This outer approximation method (known as the cutting plane method) was initially introduced by Kelley Jr (1960) in convex programming [6]. Kelley's cutting plane algorithm starts with a relaxed LP (linear programming) solution. Thereafter, it find the solution by successively adding constraints (i.e., constructed cuts) to the problem [7], pp. 463-465 and [8], pp. 316-323. The outer-approximate with increasingly tighter convex programs was extended by Tuy (1983) [9] to general nonconvex optimization problems.

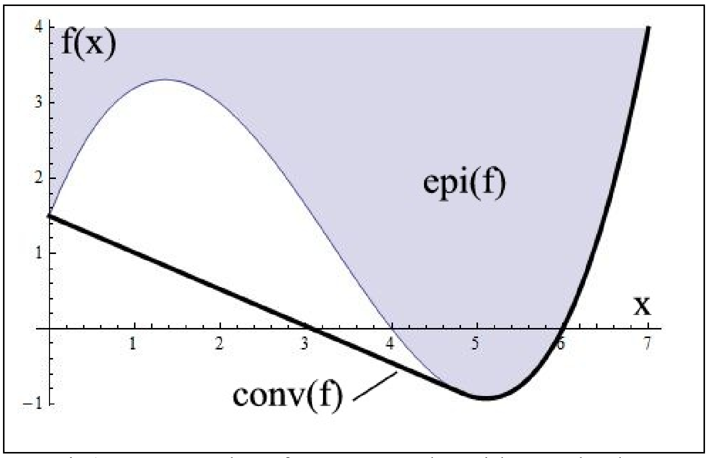

Fig 1. Convex envelope of a nonconvex polynomial over a closed convex interval.

C. Nuclear Norm

Complexity and dimensionality of the system can be expressed by means of the rank of a matrix. In [4] a low-rank matrix should correspond to different situations in statistics, system identification or control, e.g., a low-degree for a random process model, a low-order realization of a linear system. An affine rank minimization problem (RMP) consists of finding a matrix of minimum rank that satisfies a system of linear equality constraints[4].

Definition 2. The nuclear form of the $m \times n$ matrix $\mathbf{X}$ (or Schatten 1-norm, or Ky Fan r-norm) is the sum of its singular values, i.e.,

$$\|\mathbf{X}\|_* = \sum_{i=1}^{\min\{m,n\}} \sigma_i(\mathbf{X}),\ \sigma_i(\mathbf{X}) = \sqrt{\lambda_i(\mathbf{X}^T\mathbf{X})}\ .\square$$

Let the RMP [4][12] be

$$\begin{aligned}&\text{minimize } \ \operatorname{rank}(\mathbf{X})\\ &\text{subject to :}\\ &\mathcal{A}(\mathbf{X}) = \mathbf{b},\end{aligned}$$

where $\mathcal{A}: \mathbb{R}^{m\times n} \mapsto \mathbb{R}^p$ is a linear mapping. In statistics, RMP can refer to the problem of finding the least complex stochastic model according to the available observations and prior assumptions[4] [12].

Theorem 2. The convex envelope of the rank function $\phi(\mathbf{X}) = \operatorname{rank}(\mathbf{X})$ over the set of matrices with bounded norm $\mathcal{S} = \{\mathbf{X} \in \mathbb{R}^{m\times n} : \|\mathbf{X}\| \le 1\}$ is $\phi_{\text{env}}(\mathbf{X}) = \|\mathbf{X}\|_*$.□

Proof[5]: See [12], pp. 54-60 □

Since the nuclear norm is the convex envelope of rank, the problem is

$$\begin{aligned}&\text{minimize } \ \|\mathbf{X}\|_*\\ &\text{subject to :}\\ &\mathcal{A}(\mathbf{X}) = \mathbf{b}.\end{aligned}$$

D. Conjugacy Transformation [13]

Conjugacy transformation (or Legendre-Fenchel transform) associates with any function f a convex function f^* called convex conjugate. This important notion intervenes in the Lagrangian duality. It relates the dual with the primal function.[6]

Definition.3. Let the closed convex differentiable function $f: \mathbb{R}^n \mapsto (-\infty, \infty)$.

The Fenchel conjugate $f^*: \mathbb{R}^n \mapsto (-\infty, \infty)$ is [7]

$$f^*(\mathbf{y}) \triangleq \sup_{\mathbf{x}\in\mathbb{R}^n} \{\langle \mathbf{x}, \mathbf{y}\rangle - f(\mathbf{x})\}\ .\square$$

It is a generalization of the Legendre transform[8]. It expresses the maximum gap between the linear $\mathbf{x}^T\mathbf{y}$ and $f(\mathbf{x})$ [13] pp. 82-89.

The properties of the conjugate function are

- f^* is always convex, since it is the pointwise supremum of a family of convex functions of y.
- If f and f^* are convex, and their epigraph is closed convex, then $f^{**} = f^*(f^*) = f$. Therefore, the conjugacy transform is a symmetric transformation.
- If f and f^* are convex, then they satisfy the Fenchel-Young inequality

  $f(\mathbf{x}) + f^*(\mathbf{y}) \ge \langle \mathbf{x}, \mathbf{y}\rangle$ for all $\mathbf{x}, \mathbf{y}$.

Example 2. [16], pp. 72-74. Let the univariate exponential function $f(x) = e^x$ where $x \in \mathbb{R}$. If $y < 0$, the expression $yx - e^x$ is unbounded, so that $f^*(y) = +\infty$. For $y = 0$, we

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[4] Let the variance-covariance matrix $\mathbf{X} = \mathrm{E}\left[(\tilde{\mathbf{z}} - \mathrm{E}[\tilde{\mathbf{z}}])(\tilde{\mathbf{z}} - \mathrm{E}[\tilde{\mathbf{z}}])^T\right]$ of the random $\tilde{\mathbf{z}}$. In this application, the rank of $\mathbf{X}$ denotes the complexity of the stochastic model, i.e., the number of independent random variables needed to explain the variance-covariance matrix. The trade-off that we have in practice between the model complexity (i.e., $\operatorname{rank}(\mathbf{X})$) and its accuracy $f(\mathbf{X})$ is illustrated in [12].

[5] The proof of the convex envelope theorem is using the conjugate functions.

[6] On the conjugacy correspondence, see Bertsekas *et al.* [14], pp. 432-434.

[7] The domain of the conjugate function consists of $\mathbf{y} \in \mathbb{R}^n$ for which the supremum is finite, i.e. the difference is bounded above on $\operatorname{dom}(f)$.

[8] The Legendre transform for invertible gradient of f is $f^*(\mathbf{s}) = \langle \mathbf{s}, \nabla^{-1} f(\mathbf{s})\rangle - f(\nabla^{-1} f(\mathbf{s}))$. See [15].

have $\sup_x -e^x = 0$. If $y > 0$, the expression $yx - e^x$ reaches its maximum at $\hat{x} = \log_e y$. We deduce the convex conjugate $f^*(y) = y\log_e y - y$.[9]

Example 3. Let the negative entropy[10] function $f(x) = x\log x$ on $\mathrm{dom}(f) = \mathbb{R}_{++}$. The expression $yx - x\log x$ is bounded above on $\mathbb{R}_+$ for all y. Hence $\mathrm{dom}(f^*) = \mathbb{R}$. We deduce $f^*(y) = e^{y-1}$. The epigraphs of the original function and that of its convex conjugate are pictured in Fig 2.

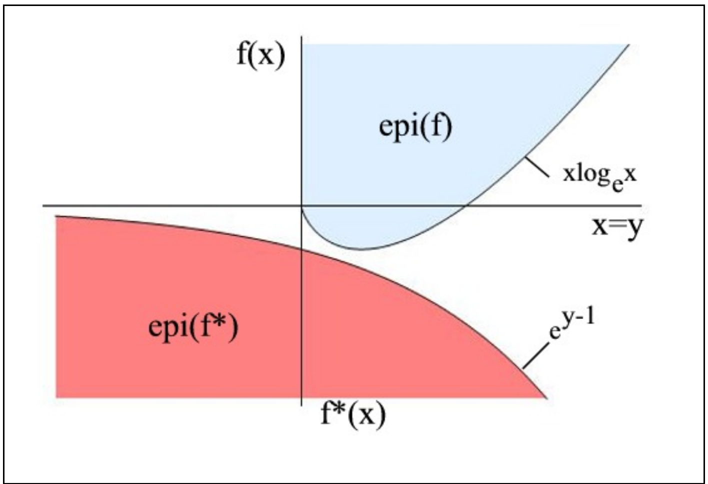

Fig 2 Convex conjugate of a negentropy function.

III. LP RELAXATIONS FOR NONCONVEX QUADRATIC POLYNOMIAL PROGRAMS

The Reformulation-Linearization Technique (RLT) by Sherali and Adams treats both discrete and continuous programming problems [5]. It is valuable for producing polyhedral outer approximations or LP relaxations for nonconvex polynomial programs having integral exponents for all nonlinear terms. RLT-LP relaxations of QP problems yield a lower bound on the global minimum [16][18]. New constraints and convex variables bounding types are introduced in [20][20] to obtain tighter lower bounds. The RLT procedure also benefits of various improvements of the implementation such as a range-reduction process, a constraint filtering technique, a new branching variable selection. Thus, filtering techniques have been proposed in [20] to accelerate the RLT search[11]. The relaxations are embedded in a convergent branch-and-bound algorithm.

A. *Nonconvex Quadratic Programming Problem* [10]

Let a nonconvex quadratic programming problem (NQP) subject to linear equality constraints and box-constrained decision variables, such as

$$\begin{aligned}
&\mathbf{NQP}(\Omega): \underset{\mathbf{x}\in\Omega\subset\mathbb{R}^n}{\text{minimize}}\ \mathbf{c}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{H}\mathbf{x}\\
&\text{subject to}:\\
&\mathbf{A}\mathbf{x} \le \mathbf{b},\\
&\mathbf{x}\in\Omega \equiv \left\{\mathbf{x}: x_j \in \left[x_j^L, x_j^U\right],\ j = 1,\ldots,n\right\},
\end{aligned}$$

where $\mathbf{x}\in\mathbb{R}^n$, $\mathbf{c}\in\mathbb{R}^n$ and $\mathbf{b}\in\mathbb{R}^m$. $\mathbf{H}$ is an $n\times n$ indefinite symmetric matrix, $\mathbf{A}$ is the $m\times n$ matrix of coefficients, and where the hyper-rectangle Ω defines finite lower and upper bounds on the variables, with $x_j^L < x_j^U$, $\forall j = 1,\ldots,j$. All the linear $m+2n$ inequality constraints, can be expressed by

$$\mathbf{G}_i\mathbf{x} \equiv \sum\nolimits_{k=1}^n G_{ik}x_k \le g_i,\ i = 1,\ldots,m+2n.$$

Rewriting the NQP problem we have

$$\begin{aligned}
&\mathbf{NQP}: \underset{\mathbf{x}\in\Omega\subset\mathbb{R}^n}{\text{minimize}}\ \sum_{k=1}^n c_k x_k + \frac{1}{2}\sum_{k=1}^n\sum_{l=1}^n h_{kl}x_k x_l\\
&\text{subject to}: \qquad (1)\\
&g_i - \mathbf{G}_i\mathbf{x} \ge 0,\ i = 1,\ldots,m+2n.
\end{aligned}$$

B. *Reformulation-Linearization Technique*

The reformulation-linearization technique (RLT) consists in the two following phases, the reformulation and the convexification phases

- In the reformulation phase, the constraints the constraints in (1) are replaced with a pairwise product such as $(g_i - \mathbf{G}_i\mathbf{x})(g_j - \mathbf{G}_j\mathbf{x}) \ge 0,\ 1 \le i \le j \le m+2n$.
- In the linearization/ convexification phase, each distinct quadratic term $x_k x_l$ for $1 \le k \le l \le n$ is replaced by a new RLT variable w_{kl}.

The RLT process yields the following LP relaxation of the NQP problem[12]

[9] More generally, let $f(\mathbf{x}) = \sum_{i=1}^n c_i \exp(x_i)$, $c_i > 0$. From the definition we deduce that $f^*(\mathbf{y}) = \sum_{i=1}^n \sup_{x_i\in\mathbb{R}}\{x_i y_i - c_i \exp(x_i)\}$. Then we obtain $f^*(\mathbf{y}) = \sum_{i=1}^n y_i \log_e(y_i / c_i) - \sum_{i=1}^n y_i$ for $\forall\mathbf{y} > \mathbf{0}$, i.e., the difference between the cross-entropy function and a linear function.

[10] The entropy is an index about disorder in a system (e.g., wasted energy). The negative entropy or negentropy refers to the quantity that is exported by the system to keep its own entropy at a lower level.

[11] Reduced size RLT (rRLT) [21] applies to nonconvex QP problems. rRLTs are obtained by replacing quadratic terms with linear constraints. An extension of the rRLT is proposed in [22] to general polynomial programs.

[12] The linearization of $[.]$ is denoted by $[.]_L$.

$$\underset{\mathbf{x},\mathbf{w}}{\text{minimize}} \ \sum_{k=1}^{n} c_k x_k + \frac{1}{2}\sum_{k=1}^{n} h_{kk} w_{kk} + \sum_{k=1}^{n-1}\sum_{l=k+1}^{n} h_{kl} w_{kl}$$

subject to :

$$\left[\left(g_i - \mathbf{G}_i\mathbf{x}\right)\left(g_j - \mathbf{G}_j\mathbf{x}\right)\right]_L \geq 0,\ 1 \leq i \leq j \leq m + 2n.$$

C. Branch-and-Bound Algorithm

In the branch-and-bound procedure, a list of active nodes $q \in Q_s$ is maintained at each stage s of the algorithm. Each node q corresponds to some partitioned hyperrectangle $\Omega^q \subseteq \Omega$. The RLT algorithm to solve NQP consists in the following different steps [5], pp. 263-281 and [22], pp. 675-683.

- **Step 0 : Initialization**. Set $s = 1, Q_s = \{1\}, q(s) = 1$ and $\Omega^1 \equiv \Omega$. Solve $\mathbf{LP}(\Omega^1)$ and get a solution $(\overline{\mathbf{x}}, \overline{\mathbf{w}})$ for which the objective value is $LB_1 = \mathbf{LP}(\Omega^1)$. If $\overline{\mathbf{x}}$ is feasible to $\mathbf{NQP}(\Omega)$, update $\hat{\mathbf{x}} = \mathbf{x}^1$ and $\hat{v} = \mathbf{c}^T\hat{\mathbf{x}} + (1/2)\hat{\mathbf{x}}^T\mathbf{H}\hat{\mathbf{x}}$. If $LB_1 = \hat{v}$, then STOP. Otherwise, determine a branching variable x_p. The index p is such that $p \in \arg\max\{\theta_k, k = 1,\ldots,n\}$

  where

  $$\theta_k \equiv \max\left\{0, h_{kk}\left(\overline{x}_k^2 - \overline{w}_{kk}\right)\right\} + \sum_{l=1}^{n} \max\left\{0, h_{kl}\left(\overline{x}_k\overline{x}_l - \overline{w}_{kl}\right)\right\},\ \theta_k > 0,$$

  for $k = 1,\ldots,n$. Then, GO TO STEP 1.

- **Step 1 : Partitioning**. Partition the selected active node $\Omega^{q(s)}$ into two sub-hyperrectangles. Denote the lower and upper bounds by $l^{q(s)}$ and $u^{q(s)}$ respectively. Then, the bounding interval $\left[l_p^{q(s)}, u_p^{q(s)}\right]$ is divided for x_p at a value $\overline{x}_p$, say $\left[l_p^{q(s)}, \overline{x}_p\right]$ and $\left[\overline{x}_p, u_p^{q(s)}\right]$. Replace $q(s)$ by these two new nodes and revise Q_s.

- **Step 2: Bounding**. Solve the LP relaxation for each of the two nodes. Update the incumbent solution if possible. Determine a corresponding branching variable index, as in the initialization STEP 0.

- **Step 3: Fathoming**. Fathom non improving nodes by setting $Q_{s+1} = Q_s - \{q \in Q_s : LB_q + \varepsilon \geq \hat{v}\}$ where ε denote a positive tolerance[13]. If $Q_{s+1} = \varnothing$, then STOP. Otherwise increment s by one and GO TO STEP 4.

- **Step 4: Node selection.** Select an active node $q(s) \in \arg\min\{LB_q : q \in Q_s\}$. RETURN TO STEP 1.

IV. LMI RELAXATIONS

Following the Shor's LMI formulation, [23][24] use LMI relaxations for solving nonconvex optimization problems. A hierarchy of LMI relaxations of increasing dimensions generates a monotone converging sequence of lower bounds to the global optimal solution. This section introduces to the LMI feasibility sets, SDP formulation of SOS polynomials, and illustrates these notions with a simplified engineering application, in structural optimization.

A. LMI Feasibility Sets

A linear matrix inequality (LMI) is of the canonical negative definite form [25]-[27]

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + x_1\mathbf{F}_1 + \cdots + x_n\mathbf{F}_n \prec 0,$$

where $\mathbf{F}_0, \mathbf{F}_1, \ldots, \mathbf{F}_n$ are symmetric $m \times m$ matrices (i.e. $\mathbf{F}_0, \mathbf{F}_i \in \mathbb{S}^m$, $i = 1,\ldots,n$) and $\mathbf{x} \in \mathbb{R}^n$.

LMI is equivalent to semialgebraic sets of polynomial inequalities and equations. Converting an SDP to a semialgebraic set is illustrated as follows [28]

$$\left\{\mathbf{X} \in \mathbb{S}^n : \mathbf{X} = \begin{pmatrix} 3-x & -(x+y) & 1 \\ -(x+y) & 4-y & 0 \\ 1 & 0 & -x \end{pmatrix} \succ 0\right\} \quad (2)$$

where $x, y \in \mathbb{R}$ are parameters. Determine the principal minors of $\mathbf{X}$. Cone $\mathbf{X}$ will satisfy (2) if and only if the parameters satisfy the polynomial inequalities[14]

$$\left.\begin{aligned} 3 - x > 0 \quad (a) \\ (3-x)(4-y) - (x+y)^2 > 0 \quad (b) \\ -x\left((3-x)(4-y) - (x+y)^2\right) - (4-y) > 0 \quad (c) \end{aligned}\right\} \quad (3)$$

The feasible set with curved faces (also called spectrahedra) of x and y is shown in Fig. 3.

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[13] An exact desired optimum requires $\varepsilon = 0$.

[14] For a square $n \times n$ matrix $\mathbf{X}$, then $\mathbf{X} \succ \mathbf{0}$ if and only if $\det(\mathbf{X}_k) > 0$ for all $k = 1,\ldots,n$, where $\mathbf{X}_k$ denotes the $k \times k$ principal minor submatrices. In the case of semidefinite $\mathbf{X} \succeq \mathbf{0}$ the conditions include all the minors.

The Shur complement is used to reformulate quadratic convex inequality into the LMI form.

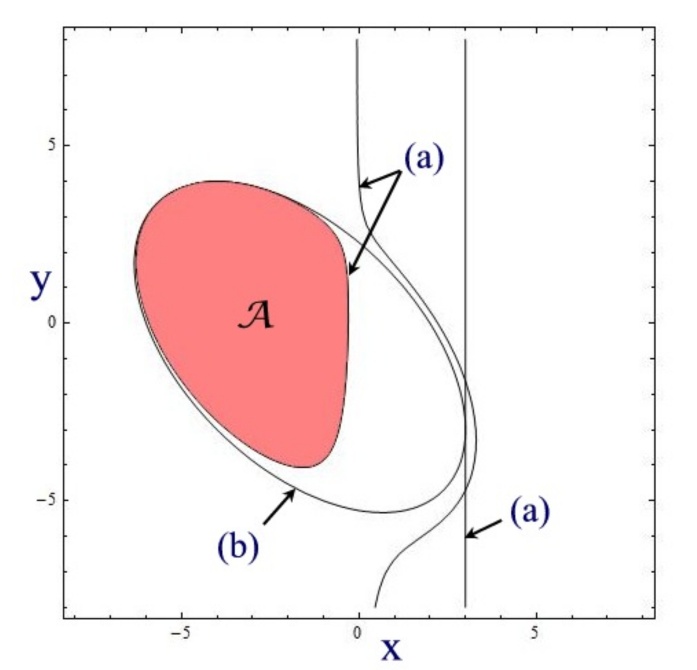

Fig. 3 . Example of a semialgebraic set.

Lemma 1. Shur Complement. Let the Hermitian block matrix $\mathbf{A} = \begin{pmatrix} \mathbf{B} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$ be a symmetric matrix with $k \times k$ block $\mathbf{B}$ and $l \times l$ block $\mathbf{D}$. Assume that $\mathbf{B} \succ \mathbf{0}$ (i.e., positive definite). Then, we have $\mathbf{A} \succ \mathbf{0}$, if and only if $\mathbf{D} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C} \succ \mathbf{0}$.

The LMI is equivalent to n polynomial inequalities. In fact, $\mathbf{F}(\mathbf{x}) \succ 0$ if and only if all its principal minors $m_k(\mathbf{x})$ are positive. We have

$$m_k(\mathbf{x}) = \det \begin{pmatrix} F_{11}(\mathbf{x}) & \dots & F_{1k}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ F_{k1}(\mathbf{x}) & \dots & F_{kk}(\mathbf{x}) \end{pmatrix}, \; k = 1, \dots, n$$

where $F_{kl}(\mathbf{x})$ denotes the entry on *k*-th and *l*-th column of $\mathbf{F}(\mathbf{x})$.

B. SDP Formulation of SOS Polynomials

Semidefinite programming (SDP) in polynomial optimization consists in approximating a hierarchy of convex semidefinite relaxations as in Shor [29]. These relaxations can be constructed by using an SOS representation of nonnegative polynomials and the dual theory of moments. Indeed, testing whether a polynomial is nonnegative can be reduced to the existence of an equivalent sum of squares (SOS) polynomial via semidefinite programming [30].

Definition 4. Let the multivariate polynomial be the following finite linear combination of monomials

$$p(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \equiv \sum_\alpha c_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n}, \; c_\alpha \in \mathbb{R},$$

where $\alpha = (\alpha_1, \dots, \alpha_n), \alpha_i \in \mathbb{N}_0$. Recall that the total degree of a monomial $\mathbf{x}^\alpha$ is equal to $\alpha_1 + \dots + \alpha_n$ and that the total degree of the polynomial is the maximum degree of its monomials[15].

Theorem 2. The existence of an SOS decomposition of a polynomial in n variables of degree $2d$, such as $p(\mathbf{x}) = \sum_i q_i^2(\mathbf{x})$ can result from a semidefinite programming feasibility problem [30][31].

The cone of SOS polynomials has an LMI formulation. A polynomial of degree $\alpha \le 2d$ is SOS if and only if

$$p(\mathbf{x}) = \mathbf{z}^T \mathbf{Q} \mathbf{z}, \text{ with } \mathbf{Q} \geq \mathbf{0},$$

where $\mathbf{z}$ contains all monomials with degree not greater than d. The Cholesky factorization yields $\mathbf{X} = \mathbf{Q}^T\mathbf{Q}$, such that $p(\mathbf{x}) = \mathbf{z}^T\mathbf{L}^T\mathbf{L}\mathbf{z} = \sum_i (\mathbf{L}\mathbf{z})_i^2$. Then, we deduce that $p(\mathbf{x}) = \sum_{i=1}^{\operatorname{rank}(\mathbf{X})} q_i^2(\mathbf{x})$.

Example 5. Let the following quartic form [30]

$$p(\mathbf{x}) = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4,$$

for which the monomial vector is $\mathbf{z} = (x_1^2, x_2^2, x_1 x_2)^T$. We have

$$\begin{aligned} p(\mathbf{x}) &= \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{pmatrix}^T \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{pmatrix}, \\ &= q_{11}x_1^4 + q_{22}x_2^4 + (q_{33} + 2q_{12}) x_1^2 x_2^2 \\ &\quad + 2q_{13}x_1^3 x_2 + 2q_{23}x_1 x_2^3. \end{aligned}$$

A positive semidefinite $\mathbf{Q}$ that satisfies the linear equalities

$$q_{11} = 2, \; q_{22} = 5, \; q_{33} + 2q_{12} = -1, \; 2q_{13} = 2 \text{ and } 2q_{23} = 0$$

is found by using SDP. A particular solution is

$$\mathbf{Q} = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{pmatrix} = \mathbf{L}^T\mathbf{L}, \; \mathbf{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Therefore, we get the SOS decomposition

[15] Special cases are homogeneous forms, where the monomials have the same total degree d. The polynomial is homogeneous of degree d, since $p(\lambda \mathbf{x}) = \lambda^d p(\mathbf{x})$.

$$p(\mathbf{x}) = \frac{1}{2}\left(2x_1^2 - 3x_2^2 + x_1x_2\right)^2 + \left(x_2^2 + 3x_1x_2\right)^2.$$

C. *Truss Topology Design*

A truss topology design (TTD) problem concerns a mechanical construction made up thin elastic bars linked to each other at nodes. The construction deforms under an external load until the tensions compensate the external forces. The goal is to design a truss of a given weight that best withstand the given weight. In other words, the compliance of the truss (i.e., potential energy resulting from the deformation) with regards to the load will be put as small as possible [25] pp. 21-29 and 227-247.[16]

Suppose that TTD problem consists in N bars of length $\mathbf{l} \in \mathbb{R}^N$ and cross-sections $\mathbf{x} \in \mathbb{R}^N$ for which lower and upper bounds are imposed, i.e., $\mathbf{a} \le \mathbf{x} \le \mathbf{b}$. Let v be the total volume of the construction, we must have $\mathbf{l}^T\mathbf{x} \le v$. Let $\mathbf{f}$ the external forces and $\mathbf{d}$ the node displacements. Let the semidefinite stiffness matrix $\mathbf{A} \succeq \mathbf{0}$ be the following linear mapping $\mathbf{A}(\mathbf{x}) = \mathbf{A}_1 x_1 + \cdots + \mathbf{A}_N x_N$. At the static equilibrium of the construction loaded by $\mathbf{f}$, we must have the nonlinear equality $\mathbf{A}(\mathbf{x})\mathbf{d} = \mathbf{f}$. The objective for the TTD problem being to minimize elastic stored energy $\mathbf{f}^T\mathbf{d}$ (i.e., maximize stiffness), the standard TTD optimization problem is

$$\begin{aligned}
&\underset{\mathbf{x}\in[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^N}{\text{minimize}} \quad \mathbf{f}^T\mathbf{d} \\
&\text{subject to :} \\
&\mathbf{A}(\mathbf{x})\mathbf{d} = \mathbf{f}, \\
&\mathbf{l}^T\mathbf{x} \le v, \\
&\mathbf{A}(\mathbf{x}) \succeq \mathbf{0}.
\end{aligned} \tag{4}$$

To obtain an equivalent LMI problem, we have to operate the following successive transformations to (4): eliminate the equilibrium constraint with $\mathbf{d} = \mathbf{A}^{-1}(\mathbf{x})\mathbf{f}$, place the objective to constraints with the auxiliary variable γ, and linearize with Schur lemma. We achieve the equivalent LMI formulation

$$\begin{aligned}
&\underset{\mathbf{x}\in[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^N}{\text{minimize}} \quad \gamma \\
&\text{subject to :} \\
&\mathbf{l}^T\mathbf{x} \le v, \\
&\begin{pmatrix} \gamma & \mathbf{f}^T \\ \mathbf{f} & \mathbf{A}(\mathbf{x}) \end{pmatrix} \succ \mathbf{0}.
\end{aligned}$$

[16] The interests of SDP for structural design in engineering are presented and developed in [32], pp. 443-467.

V. CONCLUSION: LAGRANGE AND SEMIDEFINITE RELAXATIONS

Table … shows the links between the Lagrange and semidefinite relaxation.

	● Lagrange relaxation	
A **Primal POP**	⇨	**C** **Lagrangian dual**
⇩	**◀Semi-definite relaxation▶**	⇩
B **Primal SDP**	⇔	**D** **Dual SDP**
	● Duality theory	

Table … illustrates these links by using the binary QP programming problem.

● **Lagrange relaxation**

$$\begin{bmatrix} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}}\ \mathbf{x}^T\mathbf{Q}\mathbf{x} \\ \text{s.t.}\ \ x_i^2 = 1,\ i = 1,\ldots,n \end{bmatrix} \Rightarrow \underset{\mathbf{x}}{\text{minimize}}\ L(\mathbf{x},\mathbf{y}) \triangleq \mathbf{x}^T\mathbf{Q}\mathbf{x} + \sum_i y_i\left(x_i^2 - 1\right)$$

$\mathbf{X} = \mathbf{x}^T\mathbf{x}$ ⇩ ◄ **Semidefinite relaxation** ► $\mathbf{Y} = \text{diag}(y_1,\ldots,y_n)$ ⇩

$$\begin{bmatrix} \underset{\mathbf{X}\in\mathbb{S}^n}{\text{minimize}}\ \text{tr}(\mathbf{Q}.\mathbf{X}) \\ \text{s.t.}\ \ X_{ii} = 1,\ i = 1,\ldots,n \\ \mathbf{X}\succeq\mathbf{0}_n \end{bmatrix} \Leftrightarrow \begin{bmatrix} \underset{\mathbf{Y}\in\mathbb{S}^n}{\text{minimize}}\ \text{tr}(\mathbf{Y}) \\ \text{s.t.}\ \ \mathbf{Q} - \mathbf{Y}\succeq\mathbf{0}_n,\ i = 1,\ldots,n \end{bmatrix}$$

● **Duality theory**

VI. APPENDIX A - EXAMPLE TO THE RLT PROCESS

A. Problem Formulation

Let the following nonlinear quadratic problem (NQP)[17]

$$\mathbf{NQP}: \underset{\mathbf{x}\in\Omega\subset\mathbb{R}^2}{\text{minimize}}\ \ 24x_1 - x_1^2 - x_2^2$$

subject to :

$$-3x_1 + 4x_2 \le 24,$$

$$3x_1 + 8x_2 \le 120,$$

where $\mathbf{x} = (x_1, x_2)^T$ and $\Omega = \left[x_1^L, x_1^U\right]\times\left[x_2^L, x_2^U\right]$. Using RLT, NQP is reformulated as the linearized LP:

$$\mathbf{LP}(\Omega)\ \underset{\mathbf{x}\in\mathbb{R}^2,\mathbf{w}\in\mathbb{R}^3}{\text{minimize}}\ \ 24x_1 - w_{11} - w_{22}$$

subject to :

$$G1 \equiv \left(x_1^L\right)^2 + w_{11} - 2x_1^L x_1 \ge 0,$$

$$G2 \equiv -x_1^L x_1^U - w_{11} + x_1^L x_1 + x_1^U x_1 \ge 0,$$

$$G3 \equiv x_1^L x_2^L + w_{12} - x_2^L x_1 - x_1^L x_2 \ge 0,$$

$$G4 \equiv -x_1^L x_2^U - w_{12} + x_2^U x_1 + x_1^L x_2 \ge 0,$$

$$G5 \equiv \left(x_1^U\right)^2 + w_{11} - 2x_1^U x_1 \ge 0,$$

$$G6 \equiv -x_2^L x_1^U - w_{12} + x_2^L x_1 + x_1^U x_2 \ge 0,$$

$$G7 \equiv x_1^U x_2^U + w_{12} - x_2^U x_1 - x_1^U x_2 \ge 0,$$

$$G8 \equiv \left(x_2^L\right)^2 + w_{22} - 2x_2^L x_2 \ge 0,$$

$$G9 \equiv -x_2^L x_2^U - w_{22} + x_2^L x_2 + x_2^U x_2 \ge 0,$$

$$G10 \equiv \left(x_2^U\right)^2 + w_{22} - 2x_2^U x_2 \ge 0,$$

$$G11 \equiv -24x_1^L + 3w_{11} - 4w_{12} + 24x_1 - 3x_1^L x_1 + 4x_1^L x_2 \ge 0,$$

$$G12 \equiv -120x_1^L - 3w_{11} - 8w_{12} + 120x_1 + 3x_1^L x_1 + 8x_1^L x_2 \ge 0,$$

$$G13 \equiv 24x_1^U - 3w_{11} + 4w_{12} - 24x_1 + 3x_1^U x_1 - 4x_1^U x_2 \ge 0,$$

$$G14 \equiv 120x_1^U + 3w_{11} + 8w_{12} - 120x_1 - 3x_1^U x_1 - 8x_1^U x_2 \ge 0,$$

$$G15 \equiv -24x_2^L + 3w_{12} - 4w_{22} - 3x_2^L x_1 + 24x_2 + 4x_2^L x_2 \ge 0,$$

$$G16 \equiv -120x_2^L - 3w_{12} - 8w_{22} + 3x_2^L x_1 + 120x_2 + 8x_2^L x_2 \ge 0,$$

$$G17 \equiv 24x_2^U - 3w_{12} + 4w_{22} + 3x_2^U x_1 - 24x_2 - 4x_2^U x_2 \ge 0,$$

$$G18 \equiv 120x_2^U + 3w_{12} + 8w_{22} - 3x_2^U x_1 - 120x_2 - 8x_2^U x_2 \ge 0,$$

$$G19 \equiv 576 + 9w_{11} - 24w_{12} + 16w_{22} + 144x_1 - 192x_2 \ge 0,$$

$$G20 \equiv 2880 - 9w_{11} - 12w_{12} + 32w_{22} + 288x_1 - 672x_2 \ge 0,$$

$$G21 \equiv 14400 + 9w_{11} + 48w_{12} + 64w_{22} - 720x_1 - 1920x_2 \ge 0,$$

[17] Adapted from [22], p.683.

where $\mathbf{w} = \left(w_{11}, w_{12}, w_{22}\right)^T$. The first ten linear constraints are the linearized bound factor pairwise inequalities. The next eight linear constraints are the linearized bound-constraint factor pairwise inequalities. The last three constraints are linearized constraint factor pairwise inequalities.

B. Branch-and-Bound Resolution

Suppose the following lower and upper bounds $x_1^L = 0,\ x_1^U = 24, x_2^L = 0, x_2^U = 15$. Solving $LP\left(\Omega^1\right)$, we obtain $\left(\hat{x}_1, \hat{x}_2, \hat{w}_{11}, \hat{w}_{12}, \hat{w}_{22}\right) = \left(8, 6, 192, 48, 72\right)$ for which the objective value is $v\left(LP\left(\Omega^1\right)\right) = 92$. The solution $\hat{\mathbf{x}} = \left(8,6\right)^T$ is feasible to NQP and produces an objective value of $\hat{v} = -72$. At this stage, we can observe that $\hat{w}_{12} = \hat{x}_1\hat{x}_2$ is true, whereas $\hat{w}_{11} = 192 \neq \hat{x}_1^2 = 64$ and $\hat{w}_{22} = 72 \neq \hat{x}_1^2 = 36$ both differ. Hence, we need to split the interval for x_1 at $\hat{x}_1 = 8$ or for x_2 at $\hat{x}_2 = 6$.

Using the branching rule to decide, we obtain $\theta_1 = \max\left\{0, -\left(64 - 192\right)\right\} = 128$ and $\theta_2 = \max\left\{0, -\left(36 - 72\right)\right\} = 36$. Comparing the results, we select x_1 which achieves the best value. Then, we replace the interval Ω^1 with two sub-hyperrectangles $\Omega^2 = \left\{\mathbf{x} : x_1 \in \left[0,8\right], x_2 \in \left[0,15\right]\right\}$ and $\Omega^3 = \left\{\mathbf{x} : x_1 \in \left[8,24\right], x_2 \in \left[0,15\right]\right\}$. Thereafter, using the same procedure, we obtain results for the other steps in TABLE 1. We observe that the convergence is achieved at step 2 where $v\left(LP\left(\Omega^2\right)\right) = v\left(LP\left(\Omega^3\right)\right) = v^*$.

TABLE 1 SUCCESSIVE RELAXATIONS

Relaxation #	**Decision and Lifting Variables**					**Objective Functions**	
	x_1	x_2	w_{11}	w_{12}	w_{22}	v	$v\left(\#\right)$
$LP\left(\Omega^1\right)$	8	6	192	48	72	-72	92
$LP\left(\Omega^2\right)$	0	6	0	0	36	-36	-36
$LP\left(\Omega^3\right)$	24	6	576	144	36	-36	-36

where $\Omega^1 = \left[0,24\right]\times\left[0,15\right]$, $\Omega^2 = \left[0,8\right]\times\left[0,15\right]$ and $\Omega^3 = \left[8,24\right]\times\left[0,15\right]$.

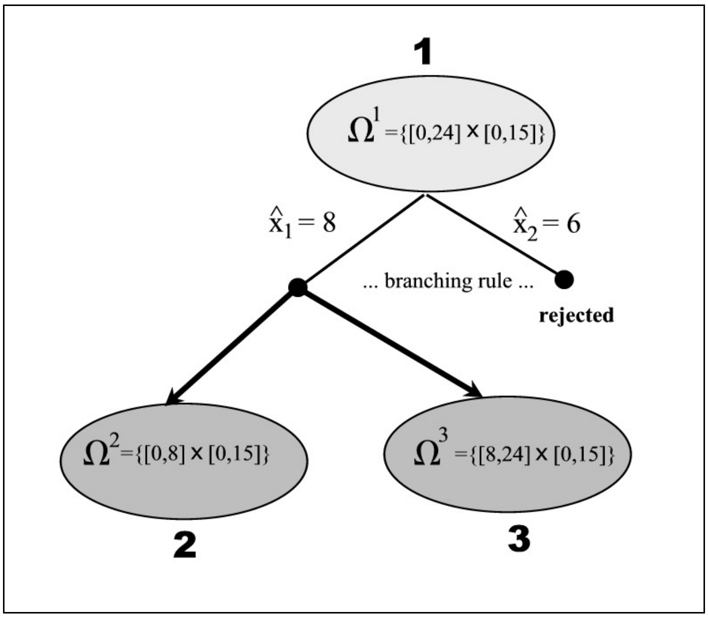

Fig. 1. Branch-and-bound decision tree.

VII. APPENDIX B - SEMIDEFINITE PROGRAMMING TO QP PROBLEMS

QP problems can be interpreted as SDP problems by using the Schur complements with regular and singular matrices. The QP problems are extended by considering an unconstrained QP, a bilinear QP and a single constraint QP[18]. The complexity of nonconvex quadratic problems is studied in [36]. It is shown that even one negative eigenvalue makes the problem NB hard.

C. Unconstrained Quadratic Optimization Problem

Let the unconstrained nonconvex QP be

$$\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}}\ \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x} + \mathbf{q}^T\mathbf{x} + r,$$

where $\mathbf{P} \in \mathbb{S}^n$. For $\mathbf{P} \succ \mathbf{0}$, the optimal value is $p^* = -\left(1/2\right)\mathbf{q}^T\mathbf{P}^{-1}\mathbf{q} + r$. More generally, we have

$$p^* = \begin{cases} -\left(1/2\right)\mathbf{q}^T\mathbf{P}^{\dagger}\mathbf{q} + r, & \text{for } \mathbf{P}\succeq\mathbf{0},\ \mathbf{q} \in \mathcal{R}\left(\mathbf{P}\right) \\ -\infty, \text{otherwise}, & \end{cases}$$

where $\mathbf{P}^{\dagger}$ is the pseudo-inverse of $\mathbf{P}$, and $\mathcal{R}\left(\mathbf{P}\right)$ denotes the range of $\mathbf{P}$.

D. Bilinear Quadratic Optimization Problem

Let the bilinear QP problem be

$$\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}}\ \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\,\mathbf{y}^T\mathbf{B}^T\mathbf{x} + \mathbf{y}^T\mathbf{C}\mathbf{y} \tag{5}$$

---

[18] This presentation is inspired from Boyd and Vandenberghe [33]. A large number of real-world applications , e.g., in engineering models, design and control can be QPs with a quadratic objective and a linear set of constraints. The properties of QPs and the different techniques for solving QPs are reviewed in [34]. The theory of nonconvex QP problems via SDPs is discussed in Nesterov *et al.* [35]. Lagrangian relaxations are used derive good approximate solutions.

Suppose that we have a regular matrix $\mathbf{A}$. The solution is $\hat{\mathbf{x}} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{y}$.

The initial QP problem (5) is rewritten as

$$\inf_{\mathbf{x}} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^T \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}.$$

The Schur complement of $\mathbf{A}$ in the partitioned matrix is $\mathbf{S} = \mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$. Using the optimal expression for $\mathbf{x}$, we find the optimal value

$$p^* = \mathbf{y}^T\left(\mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)\mathbf{y}.$$

Suppose that we have a singular matrix $\mathbf{A}$. If $\mathbf{A}\succeq\mathbf{0}$ and the range condition[19] $\mathbf{B}\mathbf{y} \in \mathcal{R}(\mathbf{A})$, then the QP problem is solvable, and the optimal value for this problem is generalized as follows

$$p^* = \mathbf{y}^T\left(\mathbf{C} - \mathbf{B}^T\mathbf{A}^{\dagger}\mathbf{B}\right)\mathbf{y}.$$

E. Single Constraint Quadratic Optimization Problem

Let the nonconvex QP be constrained with a quadratic inequality

$$\begin{aligned} &\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \;\; \mathbf{x}^T\mathbf{A}_0\mathbf{x} + 2\,\mathbf{b}_0^T\mathbf{x} + c_0 \\ &\text{subject to :} \\ &\mathbf{x}^T\mathbf{A}_1\mathbf{x} + 2\,\mathbf{b}_1^T\mathbf{x} + c_1 \le 0. \end{aligned} \tag{6}$$

where $\mathbf{A}_i \in \mathbb{S}^n$, $\mathbf{b}_i \in \mathbb{R}^n$, and $\mathbf{c}_i \in \mathbb{R}$ for $i = 0,1$. Since the quadratic terms $\mathbf{x}^T\mathbf{A}_i\mathbf{x}$ can be expressed as $\operatorname{tr}\left(\mathbf{A}\mathbf{x}\mathbf{x}^T\right)$, a new variable $\mathbf{X}$ is defined by $\mathbf{X} = \mathbf{x}\mathbf{x}^T$. Relaxing this constraint by $\mathbf{X}\succeq\mathbf{x}\mathbf{x}^T$ and using the Schur complement, the QP problem (6) is now expressed as

$$\begin{aligned} &\text{minimize} \;\; \operatorname{tr}\left(\mathbf{A}_0\mathbf{X}\right) + \mathbf{b}_0\mathbf{x} + c_0 \\ &\text{subject to :} \\ &\operatorname{tr}\left(\mathbf{A}_1\mathbf{X}\right) + \mathbf{b}_1\mathbf{x} + c_1 \le 0, \\ &\begin{pmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{x}^T & 1 \end{pmatrix} \succeq \mathbf{0}. \end{aligned}$$

The Lagrangian of problem (6) is

$$\mathcal{L}\left(\mathbf{x},\lambda\right) = \mathbf{x}^T\left(\mathbf{A}_0 + \lambda\mathbf{A}_1\right) + 2\left(\mathbf{b}_0 + \lambda\mathbf{b}_1\right)\mathbf{x} + c_0 + \lambda c_1.$$

The dual function is $g\left(\lambda\right) = \inf_{\mathbf{x}} \mathcal{L}\left(\mathbf{x},\lambda\right)$, so that

$$g\left(\lambda\right) = \begin{cases} c_0 + \lambda c_1 - \left(\mathbf{b}_0 + \lambda\mathbf{b}_1\right)^T\left(\mathbf{A}_0 + \lambda\mathbf{A}_1\right)^{\dagger}\left(\mathbf{b}_0 + \lambda\mathbf{b}_1\right), \\ \quad \text{for } \mathbf{A}_0 + \lambda\mathbf{A}_1 \succeq \mathbf{0},\, \mathbf{b}_0 + \lambda\mathbf{b}_1 \in R\left(\mathbf{A}_0 + \lambda\mathbf{A}_1\right), \\ -\infty, \;\text{otherwise.} \end{cases}$$

The dual problem and its equivalent hypograph form are

$$\begin{cases} \underset{\lambda}{\text{maximize}} \;\; g\left(\lambda\right) \\ \text{subject to :} \\ \lambda \ge 0. \end{cases} \Leftrightarrow \begin{cases} \text{maximize} \;\; t \\ \text{subject to :} \\ g\left(\lambda\right) \ge t, \\ \lambda \ge 0. \end{cases}$$

Using the Schur complement of $\mathbf{A}_0 + \lambda\mathbf{A}_1$, the dual problem is expressed as the following SDP

$$\begin{aligned} &\text{maximize} \;\; t \\ &\text{subject to :} \\ &\lambda \ge 0, \\ &\begin{pmatrix} \mathbf{A}_0 + \lambda\mathbf{A}_1 & \mathbf{b}_0 + \lambda\mathbf{b}_1 \\ \left(\mathbf{b}_0 + \lambda\mathbf{b}_1\right)^T & c_0 + \lambda c_1 - t \end{pmatrix} \succeq \mathbf{0}. \end{aligned}$$

REFERENCES

[1] C.A. Floudas, Deterministic Global Optimization: Theory, Methods and Applications, Dordrecht, NL: Kluwer Academic Publishers, 2000.

[2] M. Tawarmalani and V. Sahinidis, ''Convex extensions and envelopes of lower semi-continuous functions'', Math. Program., ser. A, vol. 93, pp. 247-263, 2002.

[3] M. Locatelli, ''Convex envelopes for quadratic and polynomial functions over polytopes'', 2010. Available at http://www.optimization-online.org/DB_FILE/2010/11/2788.pdf.

[4] B. Recht, M. Fazel, and P.A. Parrilo, "Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization", SIAM Rev., vol.52, no. 3, pp. 471-501, 2010.

[5] H.D. Sherali and W.P. Adams, A Reformulation-Linearization Technique for Solving Discrete and Continuous Nonconvex Problems, Dordrecht, NL: Kluwer Academic Publishers, 1999.

[6] J.E. Kelley Jr,, ''The cutting plane method for solving convex programs'', J.SIAM, vol. 8, no. 4, pp. 703-712, 1960.

[7] D.G. Luenberger and Y. Ye, Linear and Nonlinear Programming, 3rd ed., New York: Springer Science+Business Media, 2008, pp. 491-497.

[8] D.P. Bertsekas, Convex optimization theory: supplementary Chapter 6 on Convex Optimization Algorithms, Belmont, MA: Athena Scientific, 2014. Available at http://www.athenasc.com/convexdualitychapter.pdf.

[9] H. Tuy, ''On outer approximation methods for solving concave minimization problems'', Acta Math.Vietnam., vol. 8, no. 2, pp. 3-34, 1983.

[10] H.D. Sherali and C.H. Tuncbilek, ''A reformulation-convexification approach for solving nonconvex quadratic programming problems'', J. Global Optim., vol. 7, no. 1, pp. 1-31, 1995.

[11] M. Locatelli and F. Schoen, ''On convex envelopes for bivariate functions over polytopes'', Math. Program., ser. A, vol. 144, no 1-2, pp. 65-91, 2014. Available at http://www.optimization-online.org/DB_FILE/2009/11/2462.pdf.

[12] M. Fazel, Matrix Rank Minimization with Applications, Ph.D. Thesis, Stanford University, 2002. Available at https://faculty.washington.edu/mfazel/thesis-final.pdf.

[13] D.P. Bertsekas, Convex Optimization Theory, Belmont, MA: Athena Scientific, 2009.

[14] D.P. Bertsekas, A. Nedic, and A.E. Ozdaglar, Convex Analysis and Optimization, Belmont, MA: Athena Scientific, 2003.

[15] Y. Lucet, '' What shape is your conjugate?'', SIAM Rev., vol. 52, no. 3, pp. 505-542, 2010. Available at http://www.optimization-online.org/DB_FILE/2007/12/1863.pdf.

[16] H. Tuy, Convex Analysis and Global Optimization, Dordrecht- Boston-London: Kluwer Academic Publishers, 1998

[19] The range condition is also given by $\left(\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger}\right)\mathbf{B}\mathbf{y} = \mathbf{0}$.

[17] H.D. Sherali and C.H. Tuncbilek, "A global optimization algorithm for polynomial programming problems using a reformulation-linearization technique", J. Global Optim., vol. 2, no. 1, pp. 101-112, 1992.

[18] H.D. Sherali and C.H. Tuncbilek, "Comparison of two reformulation-linearization technique based linear programming relaxations for polynomial programming problems", J. Global Optim., vol. 10, no. 4, pp. 381-390, 1997.

[19] H.D. Sherali and C.H. Tuncbilek, "New reformulation linearization/convexification relaxations for univariate and multivariate polynomial programming problem", Oper. Res. Lett., vol. 21, no. 1, pp. 1-9, 1997.

[20] H.D. Sherali, E. Dalkiran, and L. Liberti, ''Reduced RLT representations for nonconvex polynomial programming problems'', J. Global Optim., vol. 52, no. 3, pp. 447-469, 2012.

[21] S. Cafieri, P. Hansen, L. Létocart, L. Liberti, and F. Messine, ''Reduced RLT constraints for polynomial programming'', European Workshop on Mixed Integer Nonlinear Programming (EWMINLP10), Marseille, FR, 2010.

[22] M.S. Bazaraa, H.D. Sherali, and C.M. Shetty, Nonlinear Programming: Theory and Algorithms, 3rd ed, Hoboken, NJ: John Wiley & Sons, 2006.

[23] D. Henrion and J.-B. Lasserre, "Detecting global optimality and extracting solutions in GloptiPoly", in Positive polynomials in control, D. Henrion and A. Garulli, Eds., Berlin Heidelberg: Springer Verlag, LNCIS, vol. 312, pp. 293-310, 2005.

[24] D. Henrion and J.-B. Lasserre, "Convergent relaxations of polynomial matrix inequalities and static output feedback", IEEE Trans. Automat. Control, vol. 51, no. 2, pp. 192-202, 2006.

[25] A. Ben-Tal and A. Nemirovski, Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications, Philadelphia, PA: SIAM, 2001.

[26] B. Sulikowski and W. Paszke, ''Linear Matrix Inequalities in Control'', Institute of Control and Computation Engineering, University of Zielona Gora, Poland, 2005. Available at http://www.uz.zgora.pl/~wpaszke/materialy/ts/slides_1print.pdf

[27] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, Philadelphia, PA: SIAM, 1994.

[28] P.A. Parrilo and S. Lall, ''Semidefinite programming relaxations and algebraic optimization in control'', Eur. J. Control, vol. 9, no 2-3, pp. 307-321, 2003.

[29] N.Z. Schor, ''Quadratic optimization problems'', Soviet J. Comput. Syst. Sci., vol. 25, pp. 1-11, 1987.

[30] P.A. Parrilo, "Semidefinite programming relaxations for semialgebraic problems", Math.Program, Ser.. B 96, pp. 293-320, 2003.

[31] P.A. Parrilo, Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization, Ph.D. Thesis, Pasadena, CA: California Institute of Technology, 2000. Available at http://www.mit.edu/~parrilo

[32] A. Ben and A. Nemirovski, 'Structural design'', in Handbook of Semidefinite Programming: Theory, Algorithms, and Applications, H. Wolkowicz, R. Saigal, and L. Vandenberghe, Eds., Norwell, MA: Kluwer Academic Publishers, 1998, pp. 443-467.

[33] S. Boyd and L. Vandenberghe, Convex Optimization, Cambridge, UK: Cambridge University Press, 2004.

[34] C.A. Floudas and V. Visweswaran, ''Quadratic optimization'', in Handbook of Global Optimization, R. Horst and P.M. Pardalos, Eds., Dordrecht, NL.: Kluwer Academic Publishers, pp. 217-269, 1995.

[35] Y. Nesterov, H. Wolkowicz, and Y. Ye, ''Semidefinite programming relaxations of nonconvex quadratic optimization'', in Handbook of Semidefinite Programming: Theory, Algorithms, and Applications, H. Wolkowicz, R. Saigal, and L. Vandenberghe, Eds., Norwell, MA: Kluwer Academic Publishers, 1998, pp. 361-419.

[36] P.M. Pardalos and S.A. Vavasis, ''Quadratic programming with one negative eigenvalue is NP-hard'', J. Global Optim., vol. 1, pp. 15-22, 1991.

[37] C.A. Floudas, I.G. Akrotiriankis, S. Caratzoulas, C.A. Meyer, and J. Kallrath, ''Global optimization in the 21 st century: advances and challenges'', Symposium on Computer-Aided Process Engineering, ESCAP04, Lisbon, Portugal, 2004. Available at: http://www.astro.ufl.edu/~kallrath/files/escape04.pdf.

[38] S. Caratzoulas and C.A. Floudas, ''Trigonometric convex underestimator fot the base functions in Fourier space'', J.Optim. Theory Appl., vol. 124, no. 2, pp. 339-362, 2005.

[39] M. Tawarmalani and N.V. Sahinidis, Convexification and Global Optimization in Continuous and Mixed-Integer Nonlinear Programming: Theory, Algorithms, Sofware, and Applications, Dordrecht, NL: Kluwer Academic Publishers, 2002.

[40] A.A. Keller,''Convex relaxation methods for nonconvex polynomial optimization problems'', Applied Mathematics, Computational Science and Engineering, Plenary Lecture, Varna, Bulgaria, Sept. 13-15, 2014.

[41] N.V.Sahinidis, ''Global optimization and constraint satisfaction: the branch-and-reduce approach'', in Global Optimization and Constraint Satisfaction, C. Bliek, C. Jermann, and A. Neumaier, Eds., LNCS vol. 2861, Berlin: Springer-verlag. Available at http://cepac.cheme.cmu.edu/pasilectures/sahinidis/cocos02.pdf.

[42] G.P. McCormick,''Computability of global solutions to factorable convex programs: part I-convex underestimating problems'', Math. Program., vol. 10, pp. 147-175, 1976.

[43] X. Wang and T.-S. Chang, ''An improved univariate global optimization algorithm with improved linear lower bounding functions'', J. Global Optim., vol. 8, pp. 393-411, 1996.