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Convex Relaxation Methods for Nonconvex Polynomial Optimization Problems

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EUROPMENT

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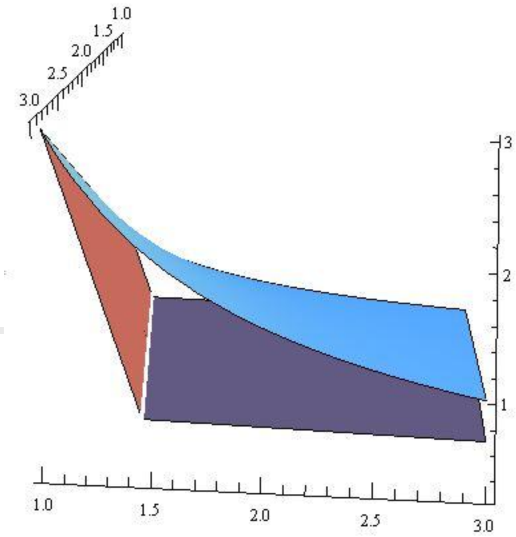
Contents

- Introduction to POPs and relaxation types
- **1**. Convexification techniques
- **2**. Reformulation-Linearization Techniques
- **3**. Semidefinite relaxations
- Conclusion on trigonometric polynomials

Introduction

Polynomial optimization problems (POPs), types of convex relaxations

FACTORABLE RELAXATION



$$\left[\begin{array}{l} z \geq \dfrac{x}{y} \ \text{ s.t.} \\ x \in \left[x^L, x^U\right],\ y \in \left[y^L, y^U\right] \end{array}\right.$$

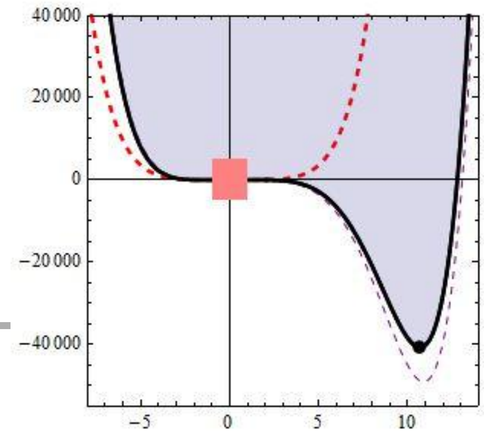
$$\left[\begin{array}{l} z\,y \geq x \ \text{ s.t.} \\ z \in \left[\dfrac{x^L y^L}{y^U}, \dfrac{x^U y^U}{y^L}\right] \\ x \in \left[x^L, x^U\right],\ y \in \left[y^L, y^U\right] \end{array}\right.$$

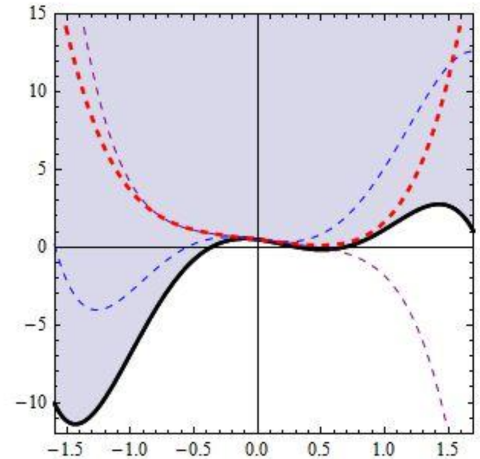
$$\left[\begin{array}{l} zy - \left(z - z^L\right)\left(y - y^U\right) \geq x, \\ zy - \left(z - z^U\right)\left(y - y^L\right) \geq x, \\ \text{s.t. } \ x \in \left[x^L, x^U\right],\ y \in \left[y^L, y^U\right]. \end{array}\right.$$

$$\left[\begin{array}{l} z \geq \left(xy^U - yx^L + x^L y^U\right) / \left(y^U\right)^2, \\ z \geq \left(xy^L - yx^U + x^U y^L\right) / \left(y^L\right)^2, \\ \text{s.t. } \ x \in \left[x^L, x^U\right],\ y \in \left[y^L, y^U\right]. \end{array}\right.$$

► **Relaxation**

EXAMPLE 1





- **Nonconvex POP**

$$\min_{x\in\mathbb{R}} \text{imize } p(x) \equiv \frac{1}{2} - x - 4x^2 + 7x^3 + \frac{1}{2}x^4 - 2x^5 + \frac{3}{20}x^6$$

subject to :

$$-8 \le x \le 14$$

▶ 2 contraints

Reformulated convex POP

$$\min_{x\in\mathbb{R},\mathbf{y}\in\mathbb{R}^5,} \text{imize } q(x,\mathbf{y}) \equiv \frac{1}{2} - y_1 - 4y_2 + 7y_3 + \frac{1}{2}y_4 - 2y_5 + \frac{3}{20}x^6$$

subject to :

$$y_1 - x = 0,$$
$$y_2 - y_1 x = 0,$$
$$y_3 - y_2 x = 0,$$
$$y_4 - y_3 x = 0,$$
$$y_5 - y_4 x = 0,$$
$$-8 \le x \le 14,$$
$$(-8, 64, -512, 4096, -32768)^T \le \mathbf{y} \le (14, 196, 2744, 38416, -32768)^T$$

▶ 14 contraints

- **Global minimum**

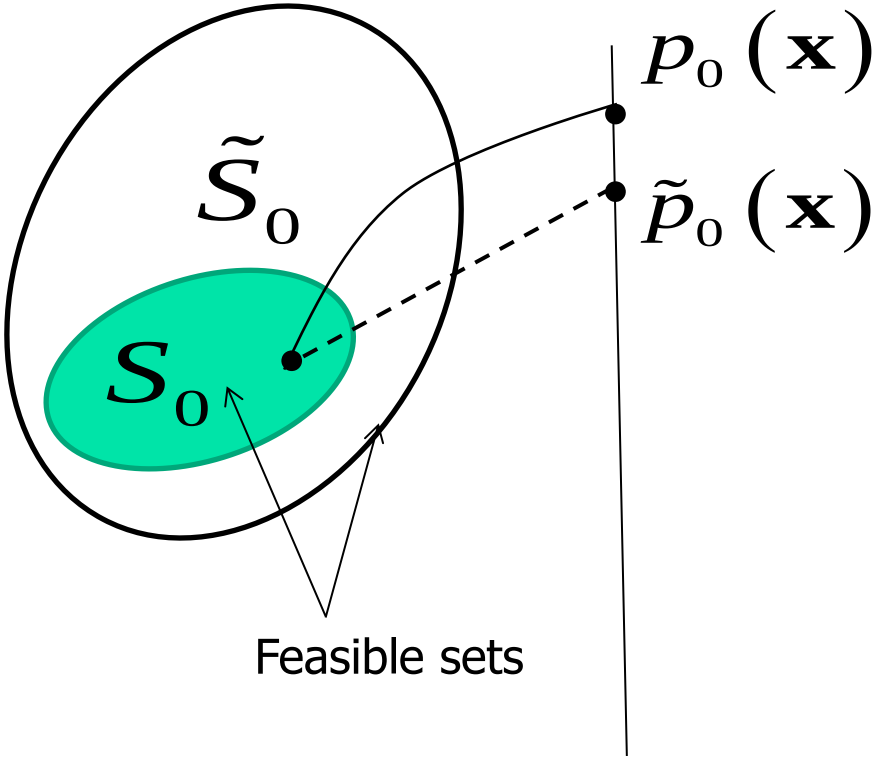
$$\hat{x} = 10.7074$$

$$q(\hat{x}, \hat{\mathbf{y}}) = p(\hat{x}) = -40,740$$

LAGRANGIAN RELAXATION

- **Inequality constrained optimization**

$$\begin{cases} \mathcal{P}_0 : \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad p_0(\mathbf{x}) \\ \text{subject to :} \\ p_j(\mathbf{x}) \le 0, \quad j = 1,\ldots,m \end{cases}$$



- **Lagrange relaxation**

$$\tilde{\mathcal{P}}_0 : \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad L(\mathbf{x},\mathbf{y}) = p_0(\mathbf{x}) + \sum_{i=1}^{m} y_i p_i(\mathbf{x})$$

αBB METHOD

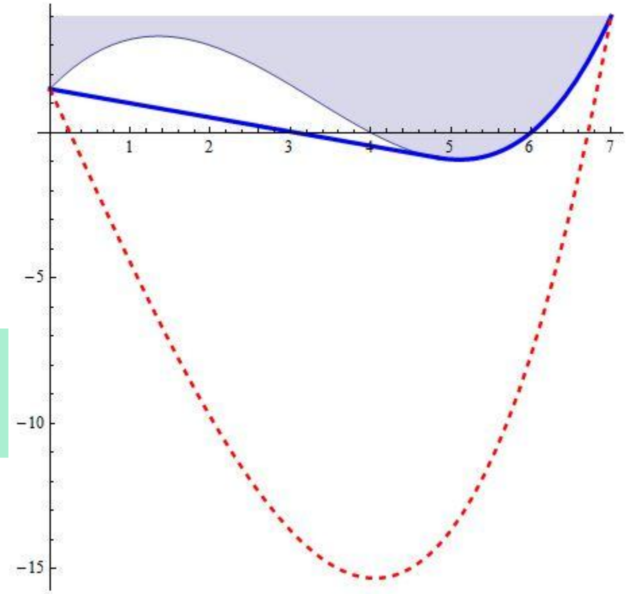
- Let a 4*th* degree polynomial function

$$p(x) = 1.5 + 2.8821 - 1.2774x^2 + 0.0955x^3 + 0.0051x^4$$

where $x \in [0,7]$.

- Define $L(x) = p(x) + \alpha(x - x^L)(x - x^U)$

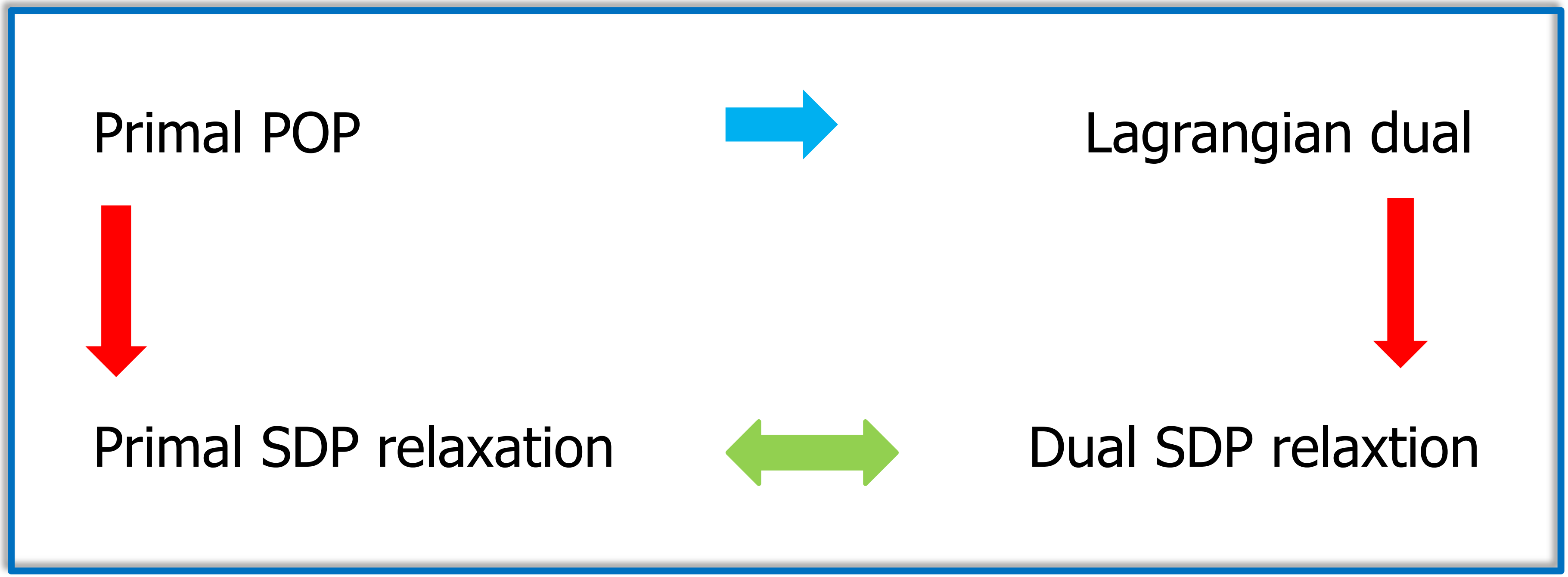
► **make** $L(x)$ **convex by overpowering the nonconvexities of** $p(x)$



Conclusion

● Lagrange and SDP relaxations; ● polynomial/bilinear matrix inequality (PMI/BMI); ● trigonometric convex underestimators

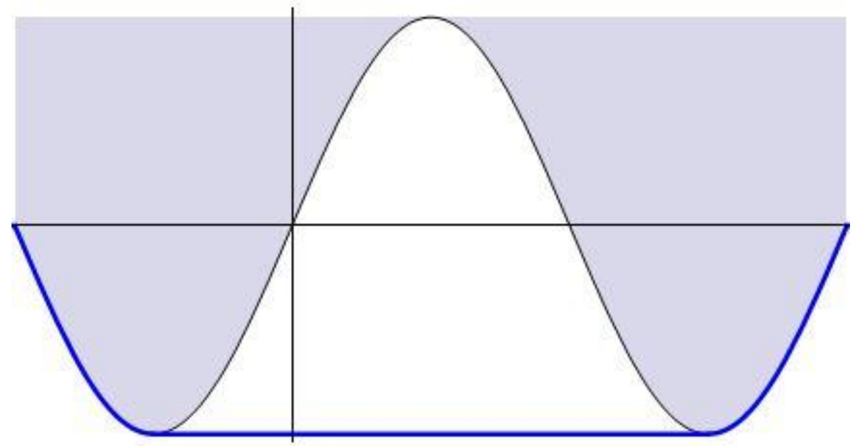
LAGRANGE AND SDP RELAXATIONS



SINE FUNCTION: CONVEX ENVELOPE

$$p(x)=\sin(x) \text{ s.t. } x\in\left[x^L,x^U\right]$$

- **Convex envelope**



$$p_{env}(x)=\begin{cases}\sin(x), & x\in\left[x^L,x_{\min}^1\right]\\ -1, & x\in\left[x_{\min}^1,x_{\min}^2\right]\\ \sin(x), & x\in\left[x_{\min}^2,x^U\right]\end{cases}$$

2 minima which value is -1 at

and $x=3\pi/2$

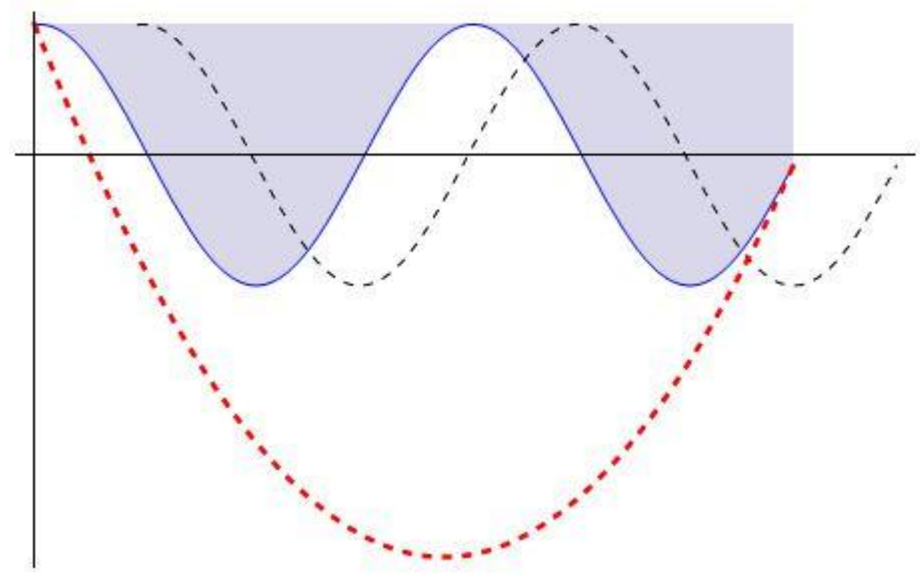
$x=-\pi/2$

2 global minima which value is -1 at: $x\in\{-\pi/2,3\pi/2\}$

$x=-\pi/2$

SINE FUNCTION: TRIGONOMETRIC CONVEX UNDERESTIMATOR

$$p(x) = \sin(x + 1.5),\ x \in [0, 10.984]$$



- **Origin**: Caratzoulas & Floudas (2005)

- The **functional form** for the function $p(x) = \alpha \sin(x + s)$ with $x \in [x^L, x^U]$ is

$$\phi(x) = -a \sin(k(x - x_s)) + b$$

- For **this example** we obtain

$$\phi(x) = -24.91 \sin(0.0979(x + 10.11)) + 21.83$$

GENERAL TRIGONOMETRIC FUNCTION: CONVEX TRIGO. UNDERESTIMATOR

- **Example** adapted from Caratzoulas & Floudas (2005)

  - The **functional form** is

    $$p(x) = \sin(x) + \sin\left(\frac{10x}{3}\right) + \log_e(x) - 0.84x$$

    where $x \in [1.5, 12.484]$

    - The figure shows the **translated** function $p(x+1.5)$ on $[0, 10.984]$ and the convex trigonometric underestimator.