Chapter 1

OPTIMAL ECONOMIC STABILIZATION POLICY UNDER UNCERTAINTY

André A. Keller

1. Introduction

A macroeconomic model can be analyzed in an economic regulation framework, by using stochastic optimal control techniques [12][20][21][38]. This regulator concept is more suitable when uncertainty is involved [5][31]. A macroeconomic model consists of difference equations which variables are of three main types : (1) endogenous variables that describe the state of the economy, (2) control variables that are the instruments of economic policy to guide the trajectory towards an equilibrium target, and (3) exogenous variables that describe an uncontrollable environment. Given the sequence of exogenous variables over time, the dynamic optimal stabilization problem consists in finding a sequence of controls, so as to minimize some quadratic objective function [40]. The optimal control is one of the possible controllers for a dynamic system, having a linear quadratic regulator and using the Pontryagin's principle or the dynamic programming method [23]][39][44]. Stochastic disturbances may affect the coefficients (multiplicative disturbances) or the equations (additive residual disturbances), provided that the disturbances are not too great [5][48]. Nevertheless, this approach encounters difficulties when uncertainties are very high or when the probability calculus is of no help with very imprecise data. The fuzzy logic contributes to a pragmatic solution of such a problem since it operates on fuzzy numbers. In a fuzzy logic, the logical variables take continue values between 0(false) and 1(true), while the classical Boolean logic operates on discrete values of either 0 or 1. Fuzzy sets are a natural extension of crisp sets [27][53]. The most common shape of their membership functions is triangular or trapezoidal. A fuzzy controller acts as an artificial decision maker that operates in a closed-loop system in real time [34]. This controller has four components : (1) a fuzzification interface to convert crisp input data into fuzzy values, (2) a static set of "If-Then" rules, (3) a dynamic inference mechanism to evaluate which control rules are relevant, and (4) the defuzzification interface that converts the fuzzy conclusions into crisp inputs of the process and explores a fuzzy learning algorithm. This contribution is concerned with optimal stabilization policies by using dynamic stochastic systems. To regulate the economy under uncertainty, the assistance of classic stochastic controllers and fuzzy controllers are considered. The computations are carried out using using the packages $MATHEMATICA^{®}$ 5.1, *MATHEMATICA*'s FuzzyLogic 2 [26][45][51], MATLAB R2008a & Simulink 7, & Control Systems, & Fuzzy Logic 2 [47]. In this chapter, we shall examine four main points about stabilization problems with macroeconomic models : (1) the stabilization of theoretical aggregate demand -aggregate supply (AS-AD) models under stochastic shocks, (2)the stabilization of the empirical stochastic multiplier-accelerator model, (3) the interest of fuzzy control of dynamic models under stochastic shocks and (4) applications to linear Phillips' model and to the nonlinear Goodwin's model. The conclusion will be dedicated to the limits to stabilization policies as in Sørensen [44]: the uncertainties attached to the credibility of the policy authorities' commitments, to the imperfect information about the state of the economy, and to the presence of time lags in the policy making.

2. Stabilization of theoretical models under stochastic shocks

2.1 Stabilization policy in a stochastic environment

According to Brainard's ([9], [49]), simple stochastic systems of the types one instrument-one target and two instruments-one target are presented. It is assumed that the policy instruments can be adjusted with nonzero costs. Thereafter, the problem of the optimal choice of instruments is introduced by using the IS-LM model retained by Poole [37].

A one instrument-one target stochastic system. A scalar model may be represented a the stochastic reduced form

$$y = \tilde{a}x + \varepsilon,$$

where *x* denotes one instrument (i.e money supply) and *y* one target (i.e national income)and where the $\tilde{a}, \varepsilon$ are random variables (RVs) with means respectively $\bar{a}, \bar{\varepsilon}$ and variances respectively $\sigma_a^2, \sigma_\varepsilon^2$. The correlation coefficient between $\tilde{a}$ and ε is denoted by ρ. In this case, $\tilde{a}$ is a multiplicative disturbance and ε is an additive disturbance. The policy maker is assumed to chose the instrument *x* so as to keep y close to some long-run objective y^*. Using a quadratic cost function, the problem is

$$\begin{aligned} \min_x \mathbf{E}\Bigg[(y-y^*)^2\Bigg], \\ \text{s.t.} \\ y = \tilde{a}x + \varepsilon. \end{aligned} \tag{1.1}$$

After some simple algebraic manipulations and taking expectations, the problem (1.1) may be simplified as

$$\min_x \Bigg\{\sigma_a^2 x^2 + \sigma_\varepsilon^2 + 2\rho\sigma_a\sigma_\varepsilon + (\bar{a}x + \bar{\varepsilon} - y^*)^2\Bigg\}.$$

From the first order condition (FOC) $\sigma_a^2 x + \rho\sigma_a\sigma_\varepsilon + (\bar{a}x + \bar{\varepsilon} - y^*)\bar{a} = 0$, we deduce the following solution for x_u under uncertainty

$$x_u = \frac{\bar{a}(y^* - \bar{\varepsilon}) - \rho\sigma_a\sigma_\varepsilon}{\bar{a}^2 + \sigma_a^2}.$$

The corresponding expression in certainty (when $\sigma_a = \sigma_\varepsilon = 0$) is $x_c = (y^* - \bar{\varepsilon})/\bar{a}$. The comparison clearly shows the invariance of the policy to additive shocks (see [49],p.311). Assuming no additive shocks ($\sigma_\varepsilon = 0$), we obtain the pair of optimal values

$$(x_u, y) = \Bigg(\frac{\bar{a}^2}{\bar{a}^2 + \sigma_a^2}x_c, \frac{a\bar{a}(y^* - \bar{\varepsilon}}{\bar{a}^2 + \sigma_a^2} + \bar{\varepsilon}\Bigg).$$

We deduce that

$$\mathbf{E}[y] - y^* = -\frac{\sigma_a^2(y^* - \bar{\varepsilon})}{\bar{a}^2 + \sigma_a^2}.$$

The target will then always undershoot its long-run objective, as long as multiplicative shocks are present ($\sigma_a^2 \neq 0$)[1]. Since policy instruments may be difficult to adjust, it is convenient to introduce costs of adjustment and to consider the rate of change of the instrument rather than its levels. Without additive disturbances , the stabilization problem (1.1) is transformed to

$$\begin{aligned} \min_{x,\dot{x}} \int_0^\infty \Bigg(m(ax-y^*)^2 + n\dot{x}^2\Bigg)e^{-rt}dt, \\ \text{s.t.} \\ y = \tilde{a}x. \end{aligned} \tag{1.2}$$

Denoting the integrand in (1.2) by

$$H(x, \dot{x}) \equiv \Bigg(m(ax-y^*)^2 + n\dot{x}^2\Bigg)e^{-rt},$$

and solving the Euler equation

$$\frac{\partial H}{\partial x} = \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right),$$

we obtain the second order linear equation

$$n\ddot{x} - nr\dot{x} - ma(ax - y^*) = 0. \tag{1.3}$$

The optimal stationary value (with $\ddot{x} = \dot{x} = 0$) will then satisfy

$$ax^* - y^* = 0.$$

The solution of (1.3) yields

$$x - x^* = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t},$$

where A_1, A_2 are arbitrary constants, and λ_1, λ_2 the two roots of the quadratic equation $n\lambda^2 - nr\lambda - ma^2 = 0$. Due to the transversality conditions, A_2 must be set to zero to ensure that *x* converges to the stationary value x^*. The optimal adjustment path is then defined by [2]

$$x - x^* = A_1 e^{\lambda_1 t}, \texttt{ where } \lambda_1 = \frac{nr - \sqrt{(n^2r^2 + 4nma^2}}{2n} < 0.$$

A two instruments-one target stochastic system. Now, suppose that the policy maker has two instruments to reach one target. The system may be

$$y = \tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \varepsilon,$$

where the means of the RVs are $\bar{a}_i, i = 1, 2, \bar{\varepsilon}$ and the variances $\sigma_i^2, i = 1, 2, \sigma_\varepsilon^2$. Moreover, the correlation coefficients between the a_i's is denoted by γ such as

$$\gamma = \frac{\texttt{Cov}[\tilde{a}_1, \tilde{a}_2]}{\sigma_{a_1}\sigma_{a_2}}.$$

Using the same quadratic cost function as in (1.1), we have the problem

$$\begin{gathered} \min_{x_1,x_2} \mathbf{E}\left[(y - y^*)^2\right], \\ \text{s.t.} \\ y = \tilde{a_1} x_1 + \tilde{a_2} x_2 + \varepsilon. \end{gathered} \tag{1.4}$$

After some algebraic manipulations and taking the expectations, the problem (1.4) is transformed to the equivalent problem

$$\min_{x_1,x_2} \left\{ x_1^2\sigma_{a_1}^2 + x_2^2\sigma_{a_2}^2 + \sigma_\varepsilon^2 + 2\sigma_{a_1}\sigma_{a_2}\gamma x_1 x_2 + (\bar{a}_1 x_1 + \bar{a}_2 x_2 + \bar{\varepsilon} - y^*)^2 \right\}.$$

The optimality conditions are

$$x_1\sigma_{a_1}^2 + \sigma_{a_1}\sigma_{a_2}\gamma x_2 + \Big(\bar{a}_1 x_1 + \bar{a}_2 x_2 + \bar{\varepsilon} - y^*\Big)\bar{a}_1 = 0, \tag{1.5}$$

$$x_2\sigma_{a_2}^2 + \sigma_{a_1}\sigma_{a_2}\gamma x_1 + \Big(\bar{a}_1 x_1 + \bar{a}_2 x_2 + \bar{\varepsilon} - y^*\Big)\bar{a}_2 = 0. \tag{1.6}$$

The solution of (1.5-1.6) will be unique provided the following condition

$$\mathtt{det}\begin{pmatrix} \sigma_{a_1}^2 + \bar{a_1}^2 & \sigma_{a_1}\sigma_{a_2}\gamma + \bar{a}_1\bar{a}_2 \\ \sigma_{a_1}\sigma_{a_2}\gamma + \bar{a}_1\bar{a}_2 & \sigma_{a_2}^2 + \bar{a_2}^2 \end{pmatrix} \neq 0.$$

The relative intensity of the two instruments can be deduced from (1.5-1.6). We have

$$\frac{x_1}{x_2} = \frac{\sigma_{a_2}(\bar{a}_1\sigma_{a_2} - \sigma_{a_1}\gamma\bar{a}_2)}{\sigma_{a_1}(\bar{a}_2\sigma_{a_1} - \sigma_{a_2}\gamma\bar{a}_1)}.$$

If the a_i's are uncorrelated ($\gamma = 0$), we show that the relative intensity of the two instruments will vary inversely with their variances, such as

$$\frac{x_1}{x_2} = \frac{\sigma_{a_2}^2}{\sigma_{a_1}^2}\frac{\bar{a}_1}{\bar{a}_2}.$$

Considering now nonzero costs of adjustment, the simplified problem with a zero discount rate may be written

$$\begin{aligned} \min_{x_1,x_2} \int_0^\infty \Big(m(y-y^*)^2 + n_1\dot{x_1}^2 + n_2\dot{x_2}^2\Big)dt, \\ \text{s.t.} \\ y = \tilde{a_1}x_1 + \tilde{a_2}x_2 + \varepsilon, \end{aligned} \tag{1.7}$$

where m, n_1, n_2 denote the adjustment costs. Denoting the integrand in (1.7) by $H(x_1, x_2, \dot{x}_1, \dot{x}_2)$, the Euler conditions will be

$$\frac{\partial H}{\partial x_1} = \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}_1}\right) \text{ and } \frac{\partial H}{\partial x_2} = \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}_2}\right).$$

We deduce that the optimality conditions are a set of simultaneous linear differential equations

$$\begin{aligned} n_1\ddot{x}_1 - ma_1(a_1x_1 + a_2x_2 - y^*) &= 0, \\ n_2\ddot{x}_2 - ma_2(a_1x_1 + a_2x_2 - y^*) &= 0. \end{aligned}$$

The relation between the stationary values for x_1, x_2 (for $\dot{x}_i = \ddot{x}_i = 0, i = 1, 2$) is

$$a_1x_1^* + a_2x_2^* = y^*.$$

The characteristic equation is a polynomial of order 4

$$n_1 n_2 \lambda^4 + m(a_1^2 n_2 + a_2^2 n_1)\lambda^2 = 0.$$

The four roots are given by

$$\lambda_{1,2} = 0,\ \lambda_{3,4} = \pm \frac{\sqrt{m(a_2^2 n_1 + a_1^2 n_2)}}{\sqrt{n_1 n_2}}.$$

To converge to the stationary values, the solution must take the form

$$\begin{aligned} x_1 - x_1^* &= A_1 e^{\lambda_4 t}, \\ x_2 - x_2^* &= A_1 \frac{a_2 n_1}{a_1 n_2} e^{\lambda_4 t}, \end{aligned}$$

where

$$\lambda_4 = -\frac{\sqrt{m(a_2^2 n_1 + a_1^2 n_2)}}{\sqrt{n_1 n_2}}.$$

The optimal choice of instruments. Poole [37] has studied the optimal choice of monetary instruments by using a simple stochastic IS-LM model. The equations of the discrete-time model are

$$y_t - y^* = -a(r_t - r^*) + \varepsilon_t, \tag{1.8}$$
$$m_t - m^* = b(y_t - y^*) - c(r_t - r^*) + \eta_t, \tag{1.9}$$

where *m* denotes the money supply, *y* the level of production and *r* the interest rates. The variables with a star are optimal values. Equation (1.8) is the IS curve and (1.9) the LM curve. The RVs ε, η are white noises with zero means and constant variances σ_ε^2 and σ_η^2. We assume that the monetary authorities have the choice between the control of the money supply and the control of the interest rate. If the authorities are controlling the interest rate ($r_t = r^*$), we have the system

$$\begin{aligned} y_t - y^* &= \varepsilon_t, \\ m_t - m^* &= b\varepsilon_t + \eta_t. \end{aligned}$$

In this case, the stabilization is only using the IS curve, since

$$\sigma_y^2 = \mathbf{E}\left[(y_t - y^*)^2\right].$$

If the monetary authorities are controlling the money supply ($m_t = m^*$), we have the system

$$\begin{aligned} y_t - y^* &= -a(r_t - r^*) + \varepsilon_t, \\ 0 &= b(y_t - y^*) - c(r_t - r^*) + \eta_t. \end{aligned}$$

Solving for $y_t - y^*$, we find

$$y_t - y^* = \frac{1}{1+\frac{ab}{c}}\left(-\frac{a}{c}\eta_t + \varepsilon_t\right),$$

and deduce that the stabilization is using both IS and LM curves. Indeed, the variance of *y* is

$$\sigma_y^2 = \frac{c^2\sigma_\varepsilon^2 + a^2\sigma_\eta^2}{(c+ab)^2}.$$

Finally with important noise on the money supply (σ_η^2), the policy of direct control of the interest rate is better.

2.2 Optimal stabilization of stochastic systems

Standard dynamic optimal stabilization problem. Let a formal stabilization problem be expressed in discrete-time by

$$\begin{aligned} \min_x \sum_{t=1}^{T}\left(y_t' M y_t + x_t' N x_t\right),\ M, N \geq 0 \\ \text{s.t.} \\ y_t = A y_{t-1} + B x_t. \end{aligned} \tag{1.10}$$

In the quadratic cost function of the problem the *n* state vector *y* and the *m* control vector *x* are deviations from long-run desired values, the positive semi-definite $n \times n$ matrix *M* and the positive semi-definite $m \times m$ matrix *N* are costs with having values away from the desired objectives. The constraint of the problem is a first-order dynamic system [3] with an $n \times n$ matrix *A* and an $n \times m$ matrix *B* of coefficients. The objective of the policy maker is to stabilize the system close to its long-run equilibrium. To find a sequence of control variables such that the state variables y_t can move from any initial y_0 to any other state y_T , the dynamically controllable condition is given by the rank of a concatenate matrix

$$\mathtt{rank}\left[B, AB, \ldots, A^{n-1}B\right] = n.$$

The solution is a linear feedback control given by

$$x_t = R_t y_{t-1},$$

where we have

$$\begin{aligned} R_t &= -\Big(N + B'S_tB\Big)^{-1}\Big(B'S_tA\Big) \\ S_{t-1} &= M + R_t'NR_t + \Big(A + BR_t\Big)'S_t\Big(A + BR_t\Big) \\ S_T &= M \end{aligned}$$

The optimal policy is then determined according a backward recursive procedure from terminal step T to the initial conditions, such as
step T:

$$\begin{aligned} S_T &= M, \\ R_T &= -\Big(N + B'S_TB\Big)^{-1}\Big(B'S_TA\Big). \end{aligned}$$

step T-1:

$$\begin{aligned} S_{T-1} &= M + R_T'NR_T + \Big(A + BR_T\Big)'S_T\Big(A + BR_T\Big), \\ R_{T-1} &= -\Big(N + B'S_TB\Big)^{-1}\Big(B'S_TA\Big). \end{aligned}$$

...
step 1:

$$\begin{aligned} S_1 &= M + R_2'NR_2 + \Big(A + BR_2\Big)'S_2\Big(A + BR_2\Big), \\ R_1 &= -\Big(N + B'S_1B\Big)^{-1}\Big(B'S_1A\Big). \end{aligned}$$

step 0:

$$\begin{aligned} S_0 &= M + R_1'NR_1 + \Big(A + BR_1\Big)'S_1\Big(A + BR_1\Big), \\ R_0 &= -\Big(N + B'S_0B\Big)^{-1}\Big(B'S_0A\Big). \end{aligned}$$

Uncorrelated multiplicative and additive shocks. The dynamic system is now subject to stochastic disturbances with random coefficients and random additive terms to each equation. The two sets of random deviation variables are supposed to be uncorrelated [4].The problem (1.10) is transformed to the stochastic formulation

$$\min_x \mathbf{E}\Big[y_t'My_t + x_t'Nx_t\Big],\ M, N \geq 0,$$

s.t.

$$y_t = (A + \Phi)y_{t-1} + (B + \Psi)x_t + \varepsilon_t.$$

The constant $n \times n$ matrix *A* and $n \times m$ matrix *B* are the deterministic part of the coefficients. The random components of the coefficients are represented by the $n \times n$ matrix Φ and the $n \times m$ matrix Ψ. Moreover, we have the stochastic assumptions : the elements $\phi_{ijt}, \psi_{ijt},$ and ε_{it} are i.i.d. with zero mean and finite variances and covariances, the elements of Φ_t are correlated with those of Ψ_t, the matrices Φ_t and Ψ_t are uncorrelated with ε_t. The solution is a linear feedback control given by

$$x_t = R_t y_{t-1},$$

where we have

$$R_t = -\Big(N + B'S_tB + \mathbf{E}\Big[\Psi'S_t\Psi\Big]\Big)^{-1}\Big(B'S_tA + \mathbf{E}\Big[\Psi'S_t\Psi\Big]\Big),$$

$$S_{t-1} = M + R_t'NR_t + \Big(A + BR_t\Big)'S_t\Big(A + BR_t\Big) + \mathbf{E}\Big[(\Phi + \Psi R)'S_t(\Phi + \Psi R)\Big].$$

5

Correlated multiplicative and additive shocks. The assumption of non correlation in the original levels equation, will necessarily imply correlations in the deviations equations. Let the initial system be defined in levels by the first order stochastic equation

$$Y_t = (A + \Phi)Y_{t-1} + (B + \Psi)X_t + \varepsilon_t,$$

and the stationary equation

$$Y^* = AY^* + BX^*.$$

By subtracting these two matrix equations and letting $y_t \equiv Y_t - Y^*$ and $x_t \equiv X_t - X^*$, we have

$$y_t = (A + \Phi)y_{t-1} + (B + \Psi)x_t + \varepsilon'_t,$$

where the additive composite disturbance ε' denotes a correlation between the stochastic component of the coefficients and the additive disturbance. The solution to the stabilization problem takes a similar expression as in the uncorrelated case. We have the solution

$$x_t = Ry_{t-1} + p,$$

where

$$R = -\Big(N + B'SB + \mathbf{E}\Big[\Psi'S\Psi\Big]\Big)^{-1}\Big(B'SA + \mathbf{E}\Big[\Psi'S\Psi\Big]\Big),$$

$$S = M + R'NR + \Big(A + BR\Big)'S\Big(A + BR\Big) + \mathbf{E}\Big[(\Phi + \Psi R)'S(\Phi + \Psi R)\Big],$$

$$p = -\Big(N + B'SB + \mathbf{E}\Big[\Psi'S\Psi\Big]\Big)^{-1}\Big(B'k + \mathbf{E}\Big[\Psi'S\Psi\Big]\Big),$$

where k is solution to the matrix equation

$$k = (A + BR)'k + \mathbf{E}\Big[(\Lambda + \Omega R)'S\varepsilon\Big].$$

The optimal policy then consists of a feedback component R together to a fixed component p. The system will oscillate about the desired targets.

2.3 Stabilization policies with a theoretical stochastic AS-AD model

In the stochastic AS-AD model of Sørensen and Whitta-Jacobsen [44], the demand and supply shocks are random. The presentation of the condensed form of the model is followed by a compact form which consists in two curves, the aggregate demand curve and the short-run supply curve. Backward-looking and rational expectations hypothesis are alternately considered in the model. The efficiency of monetary and fiscal stabilization policies is compared under stochastic environments. Optimal stabilization policies are considered when the policy makers wish to minimize some social loss function in which the volatility of output and inflation are weighted so as to express the policy's makers aversion to strong fluctuations in inflation.

Condensed form of the model. The initial version of the deterministic model in log-linearized form (see [44]) is described by equations with naive expectations.We have

$$\begin{aligned} y_t - \bar{y} &= \alpha_1(g_t - \bar{g}) - \alpha_2(r_t - \bar{r}) + v_t, && (1.11)\\ r_t &= i_t - \pi^e_{t+1}, && (1.12)\\ i_t &= \bar{r} + \pi^e_{t+1} + h(\pi_t - \pi^*) + b(y_t - \bar{y}),\ h > 0, b \geq 0 && (1.13)\\ \pi_t &= \pi^e_t + \gamma(y_t - \bar{y}) + s_t, && (1.14)\\ \pi^e_t &= \pi_{t-1}, && (1.15) \end{aligned}$$

The equation (1.11) is the aggregate demand curve [6]It approximates the percentage deviation of the output from the trend by a linear function of the relative deviation of the variables G and ε, and of the absolute deviation of variable r. The ex ante real interest rate is defined in (1.12). According to the Taylor's rule [7] (1.13), the policy makers aim at stabilizing output its trend level and have an inflation target π^*. The parameters depend on the aversion of policy makers to inflation and output instability. The short-run aggregate supply curve [8] (1.14) shows for any given expected rate of inflation, a higher output gap is associated with a higher actual rate of inflation. The naive expectations are expressed by (1.15). In a more compact form, the AD curve is obtained from (1.11) to (1.13) and the SRAS curve from (1.14) to (1.15). We have

$$\begin{aligned} \pi_t &= \pi^* + \frac{1}{\alpha}(y_t - \bar{y} - z_t),\\ \pi_t &= \pi_{t-1} + \gamma(y_t - \bar{y}) + s_t, \end{aligned}$$

where π^* denotes the inflation target of the central bank. We also have by definition

$$\alpha \equiv \frac{\alpha_2 h}{1 + \alpha_2 b} \text{ and } z_t \equiv \frac{\alpha_1(g_t - \bar{g}) + v_t}{1 + \alpha_2 b}.$$

Expressing the model in deviations trend and desired values with $\hat{y}_t = y_t - \bar{y}$ and $\hat{\pi}_t = \pi_t - \pi^*$, and solving we have the reduced form

$$\begin{aligned} \hat{y}_t &= \beta\hat{y}_{t-1} + \beta(z_t - z_{t-1}) - \alpha\beta s_t,\\ \hat{\pi}_t &= \beta\hat{\pi}_{t-1} + \gamma\beta z_t + \beta s_t, \end{aligned}$$

where z_t and s_t are demand and supply shocks respectively. The impulse-response functions of the output and inflation gaps are shown in Fig.1.1. These functions show how output and inflation react over time to demand and supply shocks. The Fig.1.1(a) shows the adjustment of the output gap and of the inflation gap to a temporary negative demand shock. The Fig.1.1(b) shows the adjustment of the gaps to a negative supply shock.

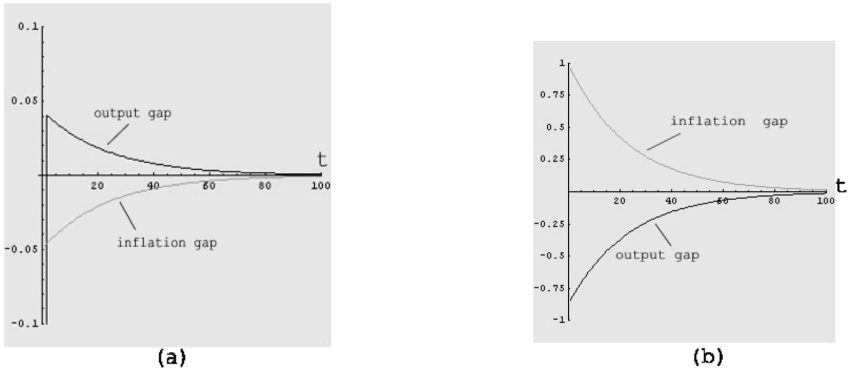

Figure 1.1. Impulse-response function of output and inflation gaps to demand (a) and supply (b) shocks

Stochastic simulations. To show the stochastic properties of the AS-AD model two types of simulations are done : one consists in demand shocks and the other supply shocks. Let us first assume the demand shock z_t be a first-order autoregressive process AR1. The stochastic AS-AD model with only demand shocks ($s_t = 0$)

$$\begin{aligned} \hat{y} &= \beta\hat{y}_{t-1} + \beta(z_t - z_{t-1}), \\ \hat{\pi} &= \beta\hat{\pi}_{t-1} + \beta\gamma z_t, \\ z_t &= \delta z_{t-1} + x_t,\ 0 \leq \delta < 1, \end{aligned}$$

where the stochastic variable is a white noise with i.i.d properties and such that $x_t \sim \mathcal{N}(0, \sigma_x^2)$. A sample of 100 observations is drawn. Let us assume now the supply shock s_t be a first-order autoregressive process AR1.The stochastic AS-AD model with only supply shocks ($z_t = 0$)is

$$\begin{aligned} \hat{y} &= \beta\hat{y}_{t-1} - \alpha\beta s_t, \\ \hat{\pi} &= \beta\hat{\pi}_{t-1} + \beta s_t, \\ s_t &= \omega s_{t-1} + c_t,\ 0 \leq \omega < 1, \end{aligned}$$

where the stochastic variable is a white noise with i.i.d properties and such that $c_t \sim \mathcal{N}(0, \sigma_x^2)$. A sample of 100 observations is drawn. The simulations are shown in Fig.1.2.

Monetary Stabilization policy under rational expectations hypothesis. Rational expectations hypothesis (REH) postulate that agents do not make any systematic forecast errors. The expectations are forward looking. The effectiveness of economic policies will be studied following Sørensen and Whitta-

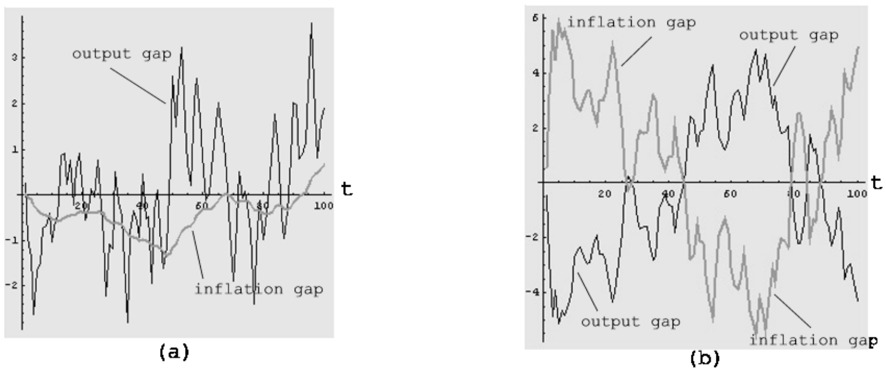

Figure 1.2. Stochastic demand (a) and supply (b) simulation

Jacobsen [44]. The model consists in three equations

$$\begin{aligned} y_t - \bar{y} &= v_t - \alpha_2(y_t - \bar{y}), && (1.16)\\ \pi_t &= \pi^e_{t,t-1} + \gamma(y_t - \bar{y}) + s_t, && (1.17)\\ r_t &= \bar{r} + h(\pi_t - \pi^*) + b(y_t - \bar{y}). && (1.18) \end{aligned}$$

Equation (1.16) is the AD curve equation, (1.17) the AS curve equation and (1.18) one possible monetary policy rule [9]. The rational expectations on prices are defined by $\pi^e_{t,t-1} = \mathbf{E}\left[\pi_t | I_{t-1}\right]$, the mathematical expectations of prices π_t conditional on all available information at this time. In this model, the stochastic demand and supply variables are white noise, i.i.d. over time, with zero means and constant variances. The solution of the model with rational expectations consists in three steps : first (step 1) solve the model for the endogenous variables y_t and π_t in terms of the exogenous variables and expectation variables $y^e_{t,t-1}$ and $\pi^e_{t,t-1}$, second (step 2) solve for the expectation variable $\pi^e_{t,t-1}$ by calculating the expected value of the expression of step 1, and finally (step 3) insert the solution of step 2 into the expression of step 1 to find the final solution for y_t and π_t. At step 1, we then have

$$\begin{aligned} y_t - \bar{y} &= \frac{v_t - \alpha_2 h s_t - \alpha_2 h(\pi^e_{t,t-1} - \pi^*)}{1 + \alpha_2(b + h\gamma)},\\ \pi_t - \bar{\pi} &= \frac{(1 + \alpha_2 b)(\pi^e_{t,t-1} - \pi^*) + (1 + \alpha_2 b)s_t + \gamma v_t}{1 + \alpha_2(b + h\gamma)}. \end{aligned}$$

At step 2, we find $\pi^e_{t,t-1} = \pi^*$. Finally at step 3, inserting the solution for rationally expected inflation, we find the final solutions

$$\begin{aligned} y_t &= \bar{y} + \frac{v_t - \alpha_2 h s_t}{1 + \alpha_2(b + h\gamma)}, \\ \pi_t &= \pi^* + \frac{(1+\alpha_2 b)s_t + \gamma v_t}{1 + \alpha_2(b + h\gamma)}. \end{aligned}$$

In these solutions, the monetary policy parameters h and b have a significative impact on the real output y_t. If the fluctuations are driven by demand shocks only (σ_s^2), the standard deviations of output and inflation are given by

$$\sigma_y = \frac{\sigma_v}{1 + \alpha_2(b + \gamma h)} \text{ and } \sigma_\pi = \frac{\gamma^2 \sigma_v}{1 + \alpha_2(b + \gamma h)}.$$

Thus, the stabilization policy will be more effective with higher policy parameters h and b. If the fluctuations are driven by supply shocks (σ_v^2), the standard deviations are

$$\sigma_y = \frac{\alpha_2 h \sigma_s}{1 + \alpha_2(b + \gamma h)} \text{ and } \sigma_\pi = \frac{(1+\alpha_2 b)\sigma_s}{1 + \alpha_2(b + \gamma h)}.$$

Optimal stabilization policies. To investigate the optimality of the monetary stabilization policy under rational expectations, we have to consider the variances $\sigma_y^2 = \mathbf{E}\left[(y_t - \bar{y})^2\right]$ and $\sigma_\pi^2 = \mathbf{E}\left[(\pi_t - \pi^*)^2\right]$. After some elementary calculations, we obtain different results from the backward-looking expectations case[10]

$$\sigma_y^2 = \frac{\sigma_v^2 + \alpha_2^2 h^2 \sigma_s^2}{(1 + \alpha_2(b + \gamma h)^2} \text{ and } \sigma_\pi^2 = \frac{\gamma^2 \sigma_v^2 + (1+\alpha_2 b)^2 \sigma_s^2}{(1 + \alpha_2(b + \gamma h)^2}.$$

Following Sørensen and Whitta-Jacobsen [44], the social loss function of the economic authorities may be expressed as $\sigma_y^2 + \kappa \sigma_\pi^2$, where κ is a preference parameter. In the general case $0 < \kappa < \infty$, there is a trade-off between stabilizing the output gap and stabilizing the inflation gap. Thus, the partial derivatives

$$\frac{\partial \sigma_y}{\partial b} = \frac{-\alpha_2^2 h \sigma_s}{(1 + \alpha_2(b + \gamma h))^2} < 0 \text{ and } \frac{\partial \sigma_\pi}{\partial b} = \frac{\alpha_2 \gamma h \sigma_s}{(1 + \alpha_2(b + \gamma h))^2} > 0$$

show that a more strongly countercyclical monetary policy, with higher values of b, will reduce the variance of output, but will increase the variance of inflation at the same time [11]

3. Stabilization of empirical models under stochastic shocks

3.1 Basic stochastic multiplier-accelerator model

Structural model. The discrete time model consists in two equations, one is the final form of output equation issued from a multiplier-accelerator model with additive disturbances, the other is a stabilization rule of proportional-derivative (PD) type. We have

$$\begin{aligned} Y_t + bY_{t-1} + cY_{t-2} &= G_t + \varepsilon_t, \\ G_t &= (g_1 - g_2)(Y_{t-1} - \bar{Y}_t) - g_2(Y_{t-1} - Y_{t-2}), \end{aligned}$$

where Y denotes the total output, G the stabilization oriented government expenditures, and ε random disturbances (zero mean, constant variance and i.i.d.) from decisions only. The policy parameters are g_1, g_2 and $\bar{Y}$ is a long run equilibrium level

Time path of output. Combining the two equations, we obtain one SDE of order 2

$$Y_t + (b - g_1)Y_{t-1} + (c - g_2)Y_{t-2} = \bar{B} + \varepsilon_t,$$

where $\bar{B}$ is a residual expression. The solution is given by

$$Y_t = \frac{\bar{B}}{1 - (b - g_1) - (c - g_2)} + C_1 r_1^t + C_2 r_2^t + \sum_{j=0}^{t-1} \frac{r_1^{j+1} - r_2^{j+1}}{r_1 - r_2} \varepsilon_{t-j},$$

where C_1, C_2 are arbitrary constants given the initial conditions, r_1, r_2 are the roots of the characteristic equation.

3.2 Stabilization of the model

Asymptotic variance of output. Provided the stability conditions are satisfied (the characteristic roots lie within the unit circle in the complex plane), the transient component will tend to zero. The system will fluctuate about the stationary equilibrium rather than converge to it. The asymptotic variance of output will be

$$\texttt{asy}\ \sigma_y^2 = \frac{1 + c + g_2}{(1 - c - g_2)\Big((1 + c + g_2)^2 - (b + g_1)^2\Big)} \sigma_\varepsilon^2.$$

The Fig.1.3 are the iso-variance and the iso-frequencies contours and show the stochastic response to changes in the parameters b and c. Attempts to stabilize the system may increase its variance. As coefficient b, c beeing held constant, the peak is shifted to a higher frequency.

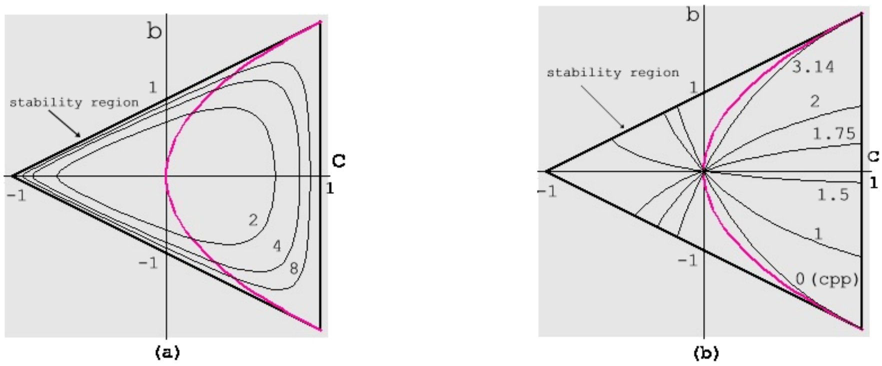

Figure 1.3. Iso-variance (a) and iso-frequencies (b) contours

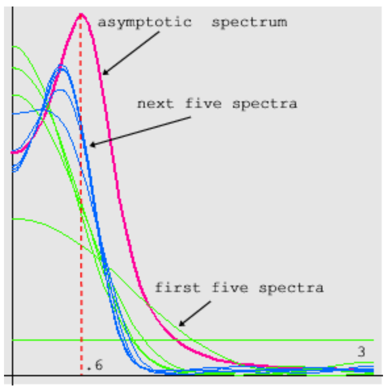

Figure 1.4. Convergence to the asymptotic spectrum

Speed of convergence. The power spectrum of y is defined by

$$f_y(\omega) = |T(\omega|^2 f_\varepsilon(\omega),$$

where the TF of the autoregressive process is given by

$$T(\omega) = \left(1 + be^{-i\omega} + ce^{-i2\omega}\right)^{-1}.$$

We also have $f_\varepsilon(\omega) = (2\pi)^{-1}\sigma_\varepsilon^2$ for serially uncorrelated ε_t. In this application, the parameters take the values $b = -1.1,\ c = .5,\ \sigma_\varepsilon^2 = 1$ as in Howrey(1967). The speed at which the spectra converge to their asymptotic values is shown in Fig.1.4.

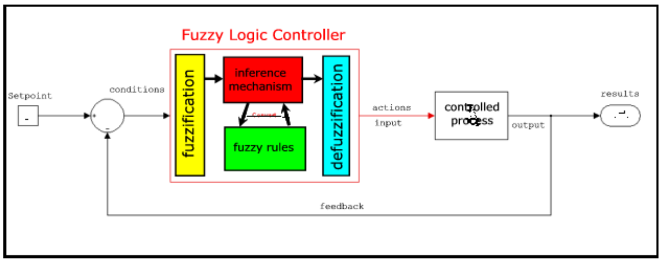

Figure 1.5. Design the fuzzy controller

Optimal policy. Policies which minimize the asymptotic variance are such $g_1^* = -b$ and $g_2^* = -c$. Then we have

$$Y_t = \bar{Y} + \varepsilon_t \text{ and } \sigma_y^2 = \sigma_\varepsilon^2.$$

output will then fluctuate about $\bar{Y}$ with variance σ_ε^2.

4. Fuzzy control of dynamic macroeconomic models

4.1 Elementary fuzzy modeling

4.1.1 Fuzzy logic controller

A fuzzy logic controller (FLC) acts as an artificial decision maker that operates in a closed-loop system in real time[34]. Fig.1.5 shows a simple control problem, keeping a desired value of a single variable. There are two conditions : the error and the derivative of the error. This controller has four components : (1) a fuzzification interface to convert crisp input data into fuzzy values, (2) a static set of "If-Then" control rules which represents the quantification of the expert's linguistic evaluation of how to achieve a good control, (3) a dynamic inference mechanism to evaluate which control rules are relevant, and (4) the defuzzification interface that converts the fuzzy conclusions into crisp inputs of the process[12]. These are the actions taken by the FLC. The process consists of three main stages : at the input stage (1) the inputs are mapped to appropriate functions, at the processing stage (2) appropriate rules are used and the results are combined, at the output stage (3) the combined results are converted to a crisp value input for the process.

4.1.2 Fuzzification

Membership functions. A membership function (MF) assigns to each element x of the universe of discourse X, a grade of membership $\mu(x)$, such that $\mu : X \mapsto [0, 1]$. The Fig.1.6 compares the crisp number to commonly used linear piesewise shapes : a triangular-shaped MF and a trapezoidal-shaped MF [13].

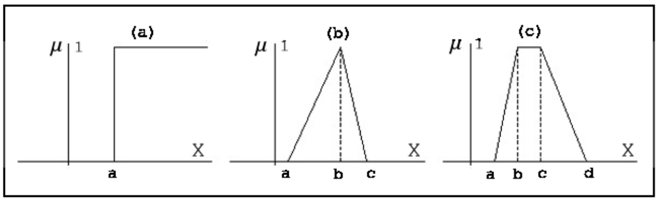

Figure 1.6. (a) Crisp set, (b) Triangular MF, (c) Trapezoidal MF

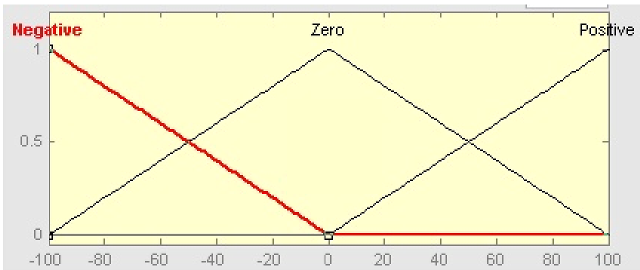

Figure 1.7. Membership functions of the two inputs and one output

The triangular MF is defined by $\mu(x) = \max\left\{\min\left\{\frac{x-a}{b-a}, \frac{c-x}{c-b}\right\}, 0\right\}$, where $a < b < c$. The trapezoidal MF is defined by $\max\left\{\min\left\{\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right\}, 0\right\}$, where $a < b < c < d$. A fuzzy set $\tilde{A}$ is then defined as a set of ordered pairs $\tilde{A} = \{x, \mu_{\tilde{A}}(x)|x \in X\}$. According to the fuzzy Zadeh operators, we have : $\mu(\tilde{A} \wedge \tilde{B}) = \min\{\mu(\tilde{A}), \mu(\tilde{B})\}$, $\mu(\tilde{A} \vee \tilde{B}) = \max\{\mu(\tilde{A}), \mu(\tilde{B})\}$ and $\mu(\neg\tilde{A}) = 1 - \mu(\tilde{A})$. The overlapping MFs of the two inputs error and change-in-error and the MF of the output control-action show the most common triangular form in Fig. 1.7. The linguistic label of these MFs are "Negative", "Zero" and "Positive" over the range $[-100, 100]$ for the two inputs and over the range $[-1, 1]$ for the output.

Fuzzy rules. Fuzzy rules are coming from expert knowledge and consist of "If-Then" statements. The linguistic rules consist of an antecedent block between "If" and "Then" and a consequent block following "Then" [14]. Let the continuous differentiable variables $e(t)$ and $\dot{e}(t)$ denote the error and the derivative of error in the simple stabilization problem of Fig. 1.5. The conditional change in error

error: control action	NL	ZE	PL
NL	NL 1	2	ZE 3
ZE	4	ZE 5	6
PL	ZE 7	8	PL 9

Figure 1.8. Fuzzy rule base 1:NL-Negative Large, ZE-Zero error,PL-Positive Large

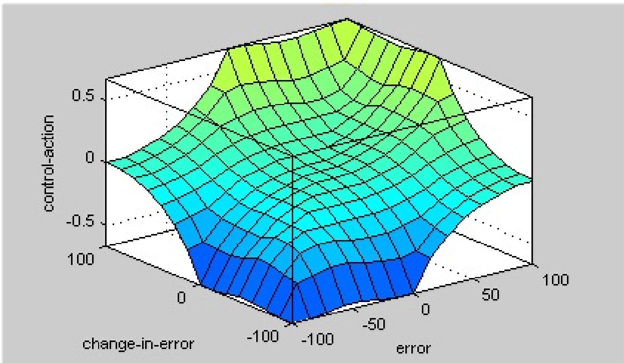

Figure 1.9. Fuzzy surface

recommendations are of the type

$$\textbf{If} \ < e, \dot{e} > \ \texttt{is} \ A \times B \ \textbf{Then} \ v \ \texttt{is} \ C, \ \texttt{where}$$
$$[A \times B](x, y) = \min[A(x), B(y)], \ x \in [-a, a], \ y \in [-b, b].$$

These FAM(Fuzzy Associative Memory)-rules [15]are those of the Fig.1.8. The commonly linguistic states of the TISO model are denoted by the simple linguistic set $A = \{NL, ZE; PL\}$. The binary input-output FAM-rules are then triples such as $(NL, NL; NL)$: "If" input e is Negative Large and $\dot{e}$ is Negative Large "Then" control action v is Negative Large. The antecedent (input) fuzzy sets are implicitly combined with conjunction "And". The control surface of this TISO control strategy is given by Fig.1.9.

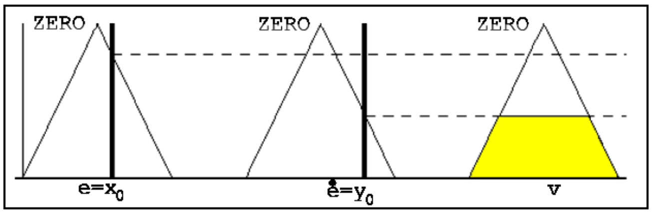

Figure 1.10. FAM influence procedure with crisp input measurements

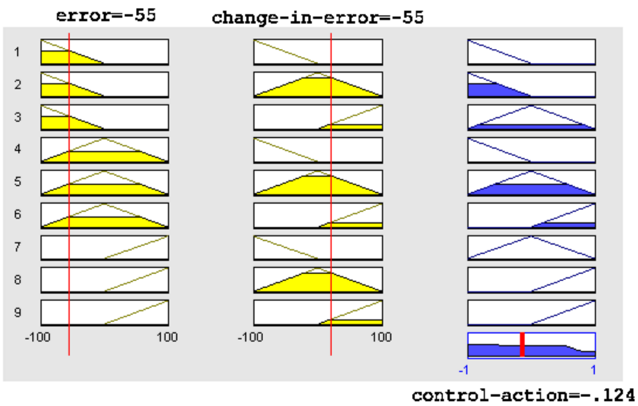

Figure 1.11. Output fuzzy set from crisp input measurements

Fuzzy inference. In Fig.1.10, the system combines logically input crisp values with minimum, since the conjunction "And" is used. Fig.1.11 produces the output set, combining all the rules of the simple control example, given crisp input values of the pair $(e, \dot{e})$.

4.1.3 Defuzzification

The fuzzy output for all rules are aggregated to a fuzzy set as in Fig.1.11. Several methods can be used to convert the output fuzzy set into a crisp value for the control-action variable v. The centroid method (or center of gravity (COG) method) is the center of mass of the area under the graph of the MF of the output set in Fig.1.11. The COG corresponds the expected value

$$v_c = \frac{\int v\mu(v)dv}{\int \mu(v)dv}.$$

In this example, $v_c = -.124$ for the pair of crisp inputs $(e, \dot{e}) = (-55, 20)$.

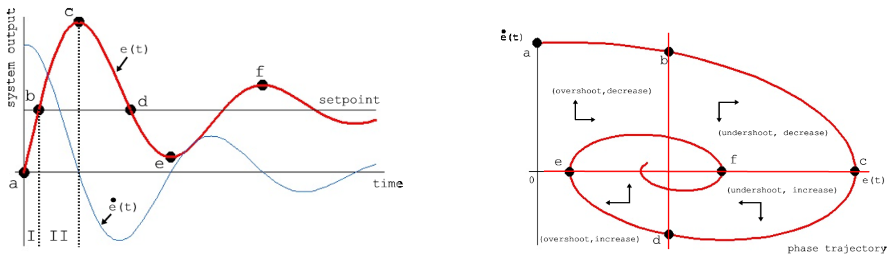

Figure 1.12. System output and fuzzy rules

4.2 TISO Mamdani fuzzy controller

Let us consider the simple control example. The fuzzy controller uses identical input fuzzy sets, namely "Negative", "Zero" and "Positive" MFs. Fig.1.11 uses the 9 numbered fuzzy rules of Fig.1.8. Let suppose the system output to follow

$$x(t) = 4 + e^{-t/5}(-4\cos t + 3\sqrt{6}\sin t),$$

as in Fig.1.12. The error is defined by $e(t) = r(t) - x(t)$, where $r(t)$ is the reference input, supposed to be constant (a setpoint)[16]. Then we have $\frac{d}{dt}e(t) = \dot{e} = -\dot{x}$. These nine rules will cover all the possible situation. According to rule I $(PL, NL; ZE)$, the system output is below the setpoint (positive error) and is increasing at this point. The controller output should then be unchanged. On the contrary, according to rule II $(NL, NL; NL)$, the system output is above the setpoint (negative error) and is increasing at this point. The controller output should then decrease the overshoot.

5. Application of fuzzy modeling to economics

Stabilization problem are considered with time-continuous multiplier-accelerator models: the linear Phillips fluctuation model and the nonlinear Goodwin's growth model [17].

5.1 The linear Phillips model

5.1.1 Structural form of the Phillips'model

The equations of the Phillips'model [1][17][35][36][43]are

$$\begin{aligned} Z(t) &= C(t) + I(t) + G(t), && (1.19)\\ C(t) &= c.Y(t) - u(t), && (1.20)\\ \frac{dI(t)}{dt} &= -\beta\Big(I(t) - v\frac{dY(t)}{dt}\Big), && (1.21)\\ \frac{dY(t)}{dt} &= -\alpha\Big(Y(t) - Z(t)\Big). && (1.22) \end{aligned}$$

All yearly variables are continuous twice-differentiable functions of time and all measured in deviation from the initial equilibrium value. The aggregate demand Z consists of consumption C, investment I and autonomous expenditures of government G in (1.19). Consumption C depends on income Y without delay and is disturbed by a spontaneous change u at time $t = 0$ in (1.20). The variable $u(t)$ is then defined by the step function $u(t) = 0,$ for $t < 0$ and $u(t) = 1$ for $t \geq 1$.The coefficient c is the marginal propensity to consume. The equation (1.21) is the linear accelerator of investment, where investment is related to the variation in demand. The coefficient v is the acceleration coefficient and β denotes the speed of response of investment to changes in production, the time constant of the acceleration lag being $\frac{1}{\beta}$ years. The equation (1.22) describes a continuous gradual production adjustment to demand. The rate of change of production Y at any time is proportional to the difference between demand and production at that time. The coefficient α is the speed of response of production to changes in demand. Simple exponential time lags are then used in this model.[18]

5.1.2 Block-diagram of the Phillips'model

The block-diagram of the whole input-output system (without PID tuning) is shown in Fig.1.13 with simulation results. The Fig.1.14 represents block-diagram of the linear multiplier-accelerator subsystem. The multiplier-accelerator subsystem shows two distinct feedbacks : the multiplier and the accelerator feedbacks.

5.1.3 System analysis of the Phillips'model

Let denote the Laplace transform of $X(t)$ by

$$\bar{X}(s) \equiv \mathcal{L}[X(t)] = \int_0^\infty e^{-st} X(t) dt.$$

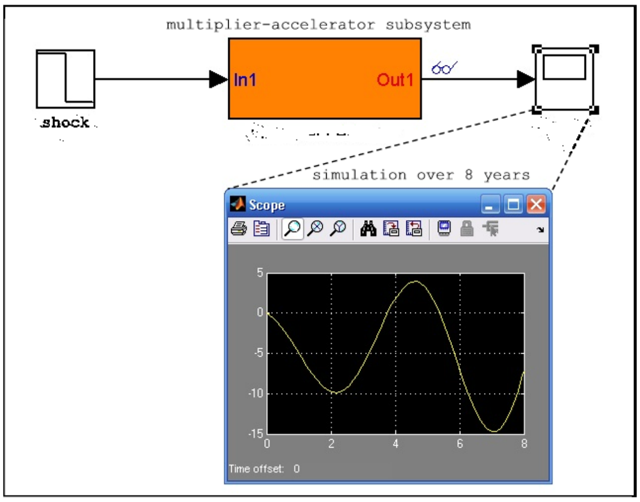

Figure 1.13. Block diagram of the system and simulation results

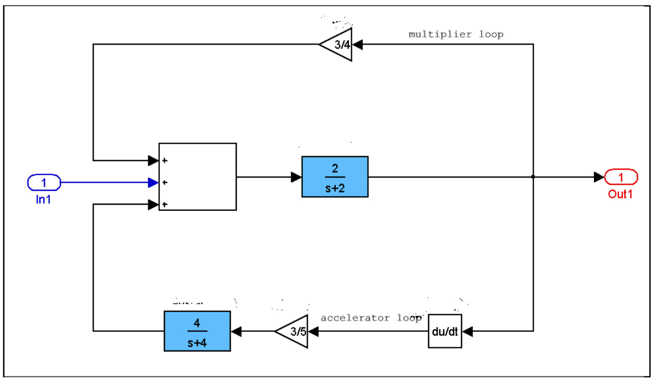

Figure 1.14. Block diagram of the linear multiplier-accelerator subsystem

Omitting the disturbance $u(t)$, the model (1.19) to (1.26) is transformed to

$$
\begin{aligned}
\bar{Z}(s) &= \bar{C}(s) + \bar{I}(s) + \bar{G}(s), && (1.23)\\
\bar{C}(s) &= c\bar{Y}(s), && (1.24)\\
s\bar{I}(s) &= -\beta\bar{I}(s) + \beta v s\bar{Y}(s), && (1.25)\\
s\bar{Y}(s) &= -\alpha\bar{Y}(s) + \alpha\bar{Z}(s). && (1.26)
\end{aligned}
$$

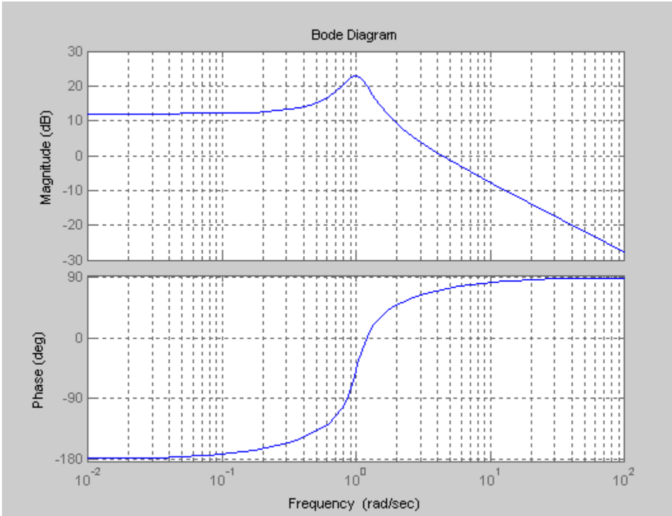

Figure 1.15. Bode diagrams of the transfer function

The transfer function (TF) of the system is deduced from the system (1.23-1.26). We have

$$H(s) \equiv \frac{\bar{Y}(s)}{\bar{G}(s)} = \frac{\alpha s + \alpha}{s^2 + \Big(\alpha(1-c) + \beta - \alpha\beta v\Big)s + \alpha\beta(1-c)}.$$

Taking a unit investment time-lag with $\beta = 1$ together with $\alpha = 4$, $c = \frac{3}{4}$ and $v = \frac{3}{5}$, we have

$$H(s) = 20\frac{s+1}{5s^2 - 2s + 5}.$$

The constant of the TF is then 4, the zero is at $s = -1$ and poles are at the complex conjugates $s = .2 \pm j$. The Bode magnitude and phase plots are shown in Fig.1.15. The magnitude expressed in decibels ($20\log_{10}$) is plotted with a log-frequency axis. The diagram shows a low frequency asymptote, a resonant peak and a decreasing high frequency asymptote. The cross-over frequency is 4 (rad/sec). To know how much a frequency will be phase-shifted, the phase (in degrees) in plotted with the a log-frequency axis. The phase cross over is near 1 (rad/sec). The TF of system is also

$$H(\mathrm{j}\omega) = \frac{20j\omega + 20}{5\omega^2 - 2j\omega + 5}.$$

When ω varies, the TF of the system is represented in Fig. 1.16 by the Nyquist diagram on the complex plane.

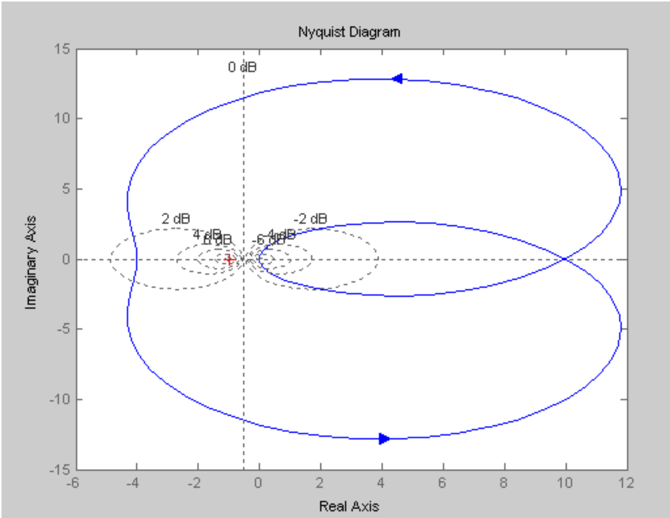

Figure 1.16. Nyquist diagram of the transfer function

5.1.4 PID control of the Phillips'model

The block-diagram of the closed-loop system with PID tuning is shown in Fig.1.17. The PID controller invokes three coefficients. The proportional gain $K_p e(t)$ determines the reaction to the current error. The integral gain $K_i = \int_0^t e(\tau)d\tau$ bases the reaction on sum of past errors. The derivative gain $K_d \frac{d}{dt} e(t)$ determines the reaction to the rate of change of error. The PID controller is a weighted sum of the three actions. A larger K_p will induce a faster response and the process will oscillate and be unstable for a excessive gain. A larger K_i eliminates steady states errors. A larger K_d decreases overshoot [**?**].[19] A PID controller is also described by the following TF in the continuous s-domain [15]

$$H_C(s) = K_p + \frac{K_i}{s} + sK_d.$$

The block-diagram of the PID controller is shown in Fig.1.18.

5.1.5 Fuzzy control of the Phillips'model

The closed-loop block-diagram of the Phillips'model is represented in Fig.1.19 with simulation results. It consists of the FLC block and of the TF of the model.The properties of the FLC controller have been described in Fig.1.5 (design of the controller), Fig.1.7 (membership functions), Fig.1.8 (fuzzy rule base), Fig.1.9 (fuzzy surface)and Fig.1.11 (output fuzzy set). The figures Fig.1.20 show the efficiency of such a stabilization policy. The range of the

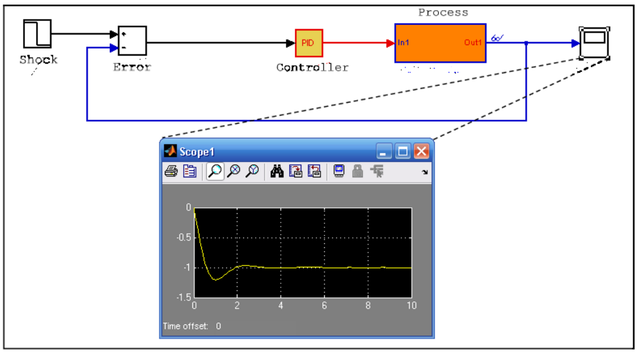

Figure 1.17. Block diagram of the closed-looped system

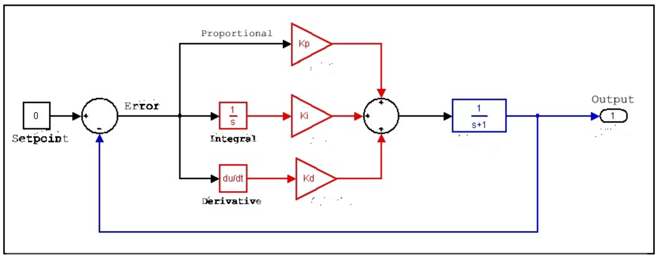

Figure 1.18. Block diagram of the PID Controller

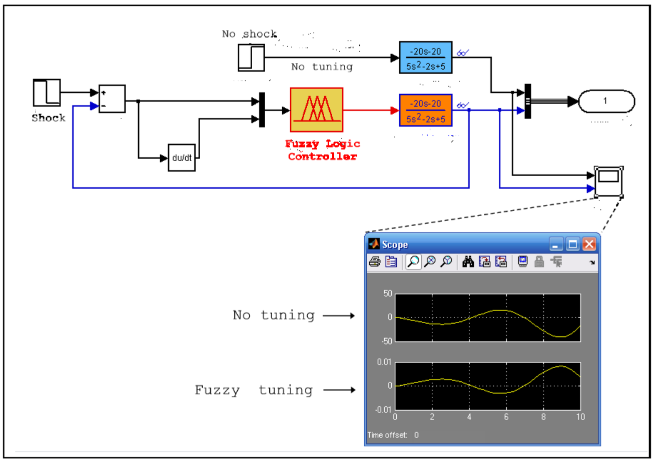

Figure 1.19. Block diagram of the Phillips model with Fuzzy Control

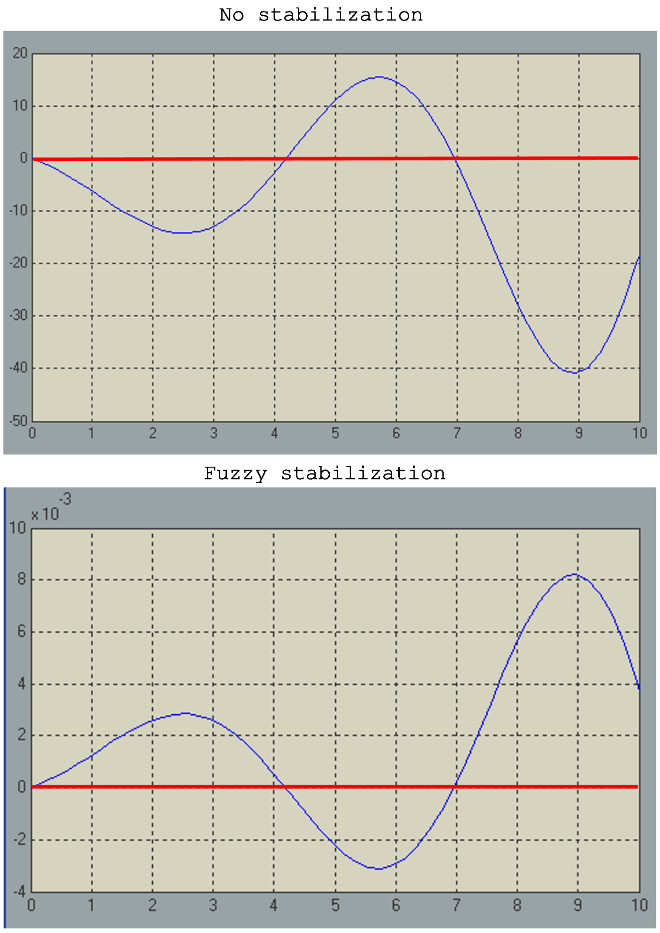

Figure 1.20. Fuzzy Stabilization of the Phillips' model

fluctuations has been notably reduced with a fuzzy control. Up to six years, the initial range $[-12, 12]$ goes to $[-3, 3]$.

5.2 The nonlinear Goodwin's model

5.2.1 Structural form of the Goodwin's model

The extended model of Goodwin [1][16][18] is a multiplier-accelerator with a nonlinear accelerator. The system is

$$\begin{aligned} Z(t) &= C(t) + I(t), && (1.27)\\ C(t) &= cY(t) - u(t), && (1.28)\\ \frac{dI(t)}{dt} &= -\beta\Big(I(t) - B(t)\Big), && (1.29)\\ B(t) &= \Phi\Big(v\frac{d}{dt}Y(t)\Big), && (1.30)\\ \frac{dY(t)}{dt} &= -\alpha\Big(Y(t) - Z(t)\Big). && (1.31) \end{aligned}$$

The aggregate demand Z in (1.27) is the sum of consumption C and total investment I [20]. The consumption function in (1.28) is not lagged on income Y. The investment (expenditures and deliveries) is determined in two stages : at the first stage, investment I in (1.29) depends on the amount of the investment decision B with an exponential lag; at the second stage the decision to invest B in (1.30) depends non linearly by Φ on the rate of change of the production Y. The equation (1.31) describes a continuous gradual production adjustment to demand. The rate of change of supply Y is proportional to the difference between demand and production at that time (with speed of response α. The nonlinear accelerator Φ is defined by

$$\Phi(\dot{Y}) = M\left(\frac{L+M}{Le^{-v\dot{Y}} + M} - 1\right),$$

where M is the scrapping rate of capital equipment and L the net capacity of the capital-goods trades. It is also subject to the restrictions

$$B = 0 \texttt{ if } \dot{Y} = 0,\ B \to L \texttt{ as } \dot{Y} \to +\infty,\ B \to -M \texttt{ as } \dot{Y} \to -\infty.$$

The graph of this function is shown in Fig.1.21.

5.2.2 Block-diagrams

The block-diagrams of the nonlinear multiplier-accelerator are described in Fig.1.22.

5.2.3 Dynamics of the Goodwin's model

The simulation results show strong and regular oscillations in Fig.1.23. The Fig.1.24 shows how a sinusoidal input is transformed by the nonlinearities. The amplitude is strongly amplified, and the phase is shifted.

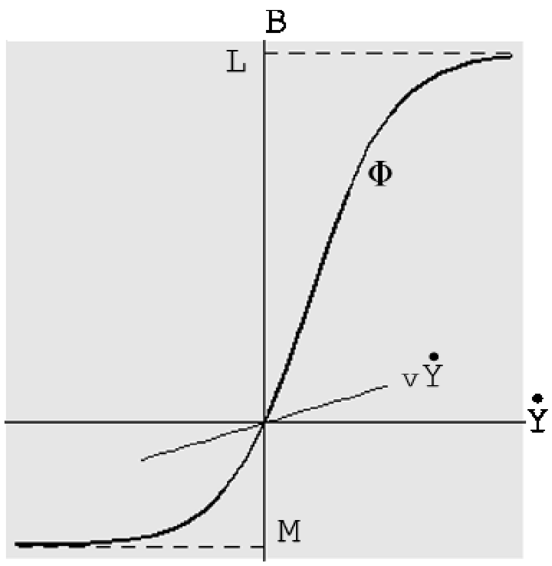

Figure 1.21. Nonlinear accelerator in the Goodwin's model

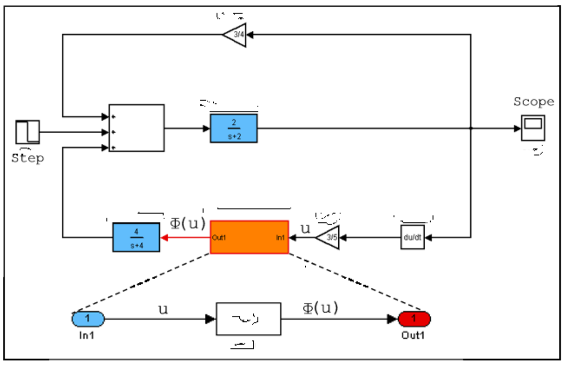

Figure 1.22. Block-diagrams of the Nonlinear accelerator

5.2.4 PID control of the Goodwin's model

The Fig.1.25 shows the block-diagram of the closed-loop system. It consists of a PID controller and of the subsystem of Fig.1.22. The Figs.1.26 show the simulation results which objective is to maintain the system at a desired level

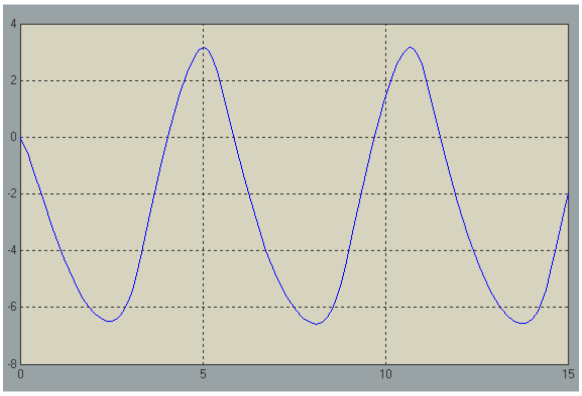

Figure 1.23. Simulation of Nonlinear accelerator

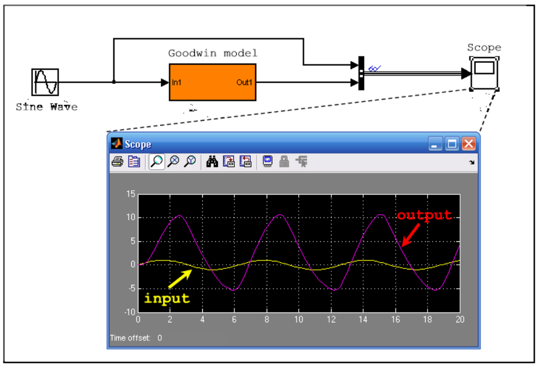

Figure 1.24. Simulation of sinusoidal input

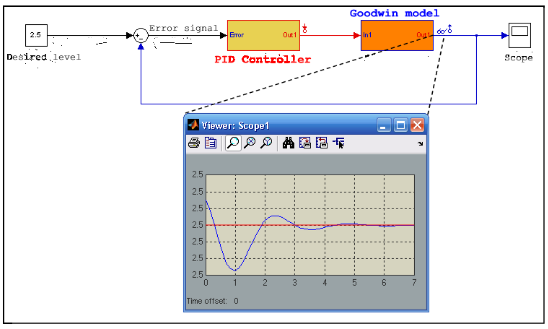

Figure 1.25. Block-diagram of the PID Controlled Goodwin's model

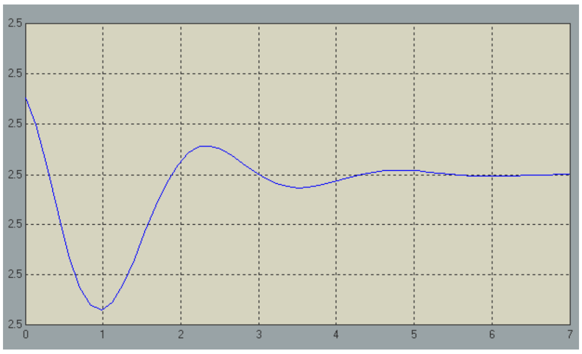

Figure 1.26. Simulation of the PID Controlled Goodwin's model

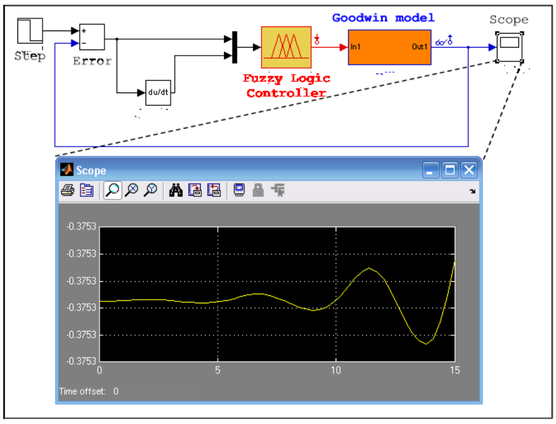

Figure 1.27. Block-diagram of the fuzzy controlled Goodwin's model

equal to 2.5. This objective is reached with oscillations within a time-period of three years Thereafter, the system is completely stabilized.

5.2.5 Fuzzy control of the Goodwin's Model

The Fig.1.27 shows the block-diagram of the controlled system. It consists of a fuzzy controller and of the subsystem of the Goodwin's model (See Fig.1.28). The FLC controller is unchanged. The simulation results in Fig.1.28 show an efficient and fast stabilization. The system is stable within five time-periods, and then fluctuates in an explosive way but restricted to an extremely close range.

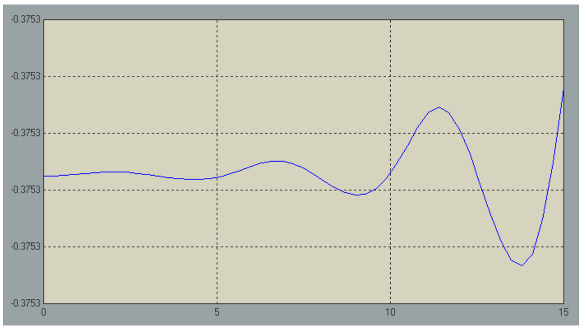

Figure 1.28. Simulation of the fuzzy controlled Goodwin's model

6. Conclusion and limits to the stabilization policies

Compared to a PID control, the simulation results of a linear and nonlinear multiplier-accelerator model show a more efficient stabilization of the economy within an acceptable time-period of few years in a fuzzy environment. Do the economic policies have the ability to stabilize the economy ? Soerensen and Whitta-Jacobsen [44] identify three major limits : the credibility of the policy authorities' commitments by rational private agents, the imperfect information about the state of the economy, and the time lags occuring in the decision making process. The effects of these limits are studied using a AS-AD model and Taylor's rule.

Credibility of policy makers' commitments. The simplified AS-AD model, such that $\gamma = 1$, without demand and supply shocks ($z_t = v_t = s_t = 0$), consists in three equations

$$\begin{aligned} \pi_t &= \pi^e_{t,t-1} + y_t - \bar{y}, && (1.32)\\ y_t - \bar{y} &= -\alpha_2(r_t - \bar{r}), && (1.33)\\ r_t &= \bar{r} + h\pi_t + b(y_t - \bar{y}), && (1.34) \end{aligned}$$

where (1.32) is the expectations-augmented Phillips curve, (1.33) the goods market equilibrium and (1.34) the Taylor's rule with zero inflation target. The central authorities are supposed to minimize a social loss function such that

$$\texttt{min}\, SL = (y_t - y^*)^2 + \kappa\pi_t^2,$$

where κ expresses the preference for price stability relative to the output stability. Thereafter, one assumption is made that the trend level of output is lower than the efficient level, due to markets imperfections. We have $y^* = \hat{y} + \omega$, $\omega > 0$, where ω denotes the markets imperfections. It follows that

$$SL = (\pi_t - \pi^e_{t,t-1} - \omega)^2 + \kappa\pi_t^2$$

. The equilibrium under Taylor rule is such that $\pi_t = \pi^e_{t,t-1} = \pi^* = 0$ and$y_t = \bar{y}$. We then have

$$\frac{dSL}{d\pi_t} = -2\omega < 0.$$

The social loss can thus be reduced when the central bank decides to reduce the output gap by a supplement inflation. The social loss SL_R in the Taylor equilibrium ($y_t = \bar{y}$ and $\pi_t = \pi^* = 0$ is $SL_R = \omega^2$. If the agents believe that the central bank will ensure price stability ($\pi^e_{t,t-1} = 0$, we then have inflation and output of

$$\pi_t = \frac{\omega}{1+\kappa} \text{ and} y_t = \bar{y} + \frac{\omega}{1+\kappa}.$$

The social loss in deviating will be $SL_C = \kappa\omega^2/(1+\kappa)$. The social welfare gain from deviating is

$$SL_R - SL_C = \frac{\omega^2}{1+\kappa}.$$

The policy maker has no incentive to implement the policy $\pi_t = \pi^* = 0$ if the private agents believe this commitment.

Imperfect information. The implications of measurement errors are studied by Sørensen and Whitta-Jacobsen [44] using the following AS-AD model. The deviations of output and inflation are given by $\hat{y}_t = y_t - \bar{y}$ and $\hat{\pi}_t = \pi_t - \pi^*$. The estimations of output and inflation deviate from the current observed values, according to the relations

$$\begin{aligned} \hat{y}^e_t &= \hat{y}_t + \mu_t, \ \mathbf{E}\Big[\mu_t\Big] = 0, \ \mathbf{E}\Big[\mu^2_t\Big] = \sigma^2_\mu, \\ \hat{\pi}^e_t &= \hat{\pi}_t + \varepsilon_t, \ \mathbf{E}\Big[\varepsilon_t\Big] = 0, \ \mathbf{E}\Big[\varepsilon^2_t\Big] = \sigma^2_\varepsilon, \end{aligned}$$

where μ_t, ε_t are the measurement errors and $\sigma^2_\mu, \sigma^2_\varepsilon$ the uncertainties in measurements.

$$\pi_t = \pi^* + \gamma(y_t - \bar{y}) \Leftrightarrow \hat{\pi}_t = \gamma \hat{y}_t, \tag{1.35}$$

$$\hat{y}_t = z_t - \alpha_2(r_t - \bar{r}), \tag{1.36}$$

$$r_t = \bar{r} + h\hat{\pi}^e_t + b y^e_t, \tag{1.37}$$

where (1.35) is the aggregate supply curve,(1.36) the good market equilibrium with demand shocks z_t and (1.37)a Taylor rule. We derive the following output gap and variance with uncorrelated stochastic variables

$$\hat{y}_t = \frac{z_t - \alpha_2 h\varepsilon_t - \alpha_2 b\mu_t}{1+\alpha_2(b+\gamma h)} \text{ and } \sigma^2_y = \frac{\sigma^2_z + \alpha^2_2 + \alpha^2_2 h^2 \sigma^2_\varepsilon + \alpha^2_2 h^2 \sigma^2_\mu}{\Big(1+\alpha_2(b+\gamma h)\Big)^2}.$$

A negative error of the output gap (μ_t) or an underestimated inflation rate (εt) will produce a positive output gap. According to the variance of the output gap, measurement errors will contribute to economic instability.

Time lags in the decision making process. Sørensen and Whitta-Jacobsen [44] also consider the consequences of time lags in the public decision process [21]. In the rewritten AS-AD model, it takes one time period (one year) for an economic change and also one year in order to affect the inflation rate. The equations are

$$\pi_{t+2} = \pi_{t+1} + \gamma(y_{t+1} - \bar{y}) + s_{t+2}, \quad (1.38)$$

$$y_{t+1} - \bar{y} = z_{t+1} - \alpha_2(i - \pi_t - \bar{r}), \quad (1.39)$$

where (1.38) is the AS curve and (1.39) is the goods market equilibrium. Using these two equations, we evaluate the effect of the nominal interest rate i_t at time t on the inflation gap two years later. We have

$$\pi_{t+2} = \pi_t + \gamma(y_t - \bar{y}) + s_{t+1} + \gamma\left(z_{t+1} - \alpha_2(i_t - \pi_t - \bar{r})\right) + s_{t+2}.$$

The expected forecast of inflation by a rational central bank will be

$$\pi^e_{t+2,t} = \pi_t + \gamma(y_t - \bar{y}) - \alpha_2\gamma(i_t - \pi_t - \bar{r}). \quad (1.40)$$

To minimize the central bank's loss function $SL = (1/2)(\pi_t - \pi^*)^2$, the authorities must choose the nominal interest rate i_t to ensure the equality $\pi^e_{t+2,t} = \pi^*$. Solving (1.40) for the nominal interest rate, we prove that the Taylor rule is an optimal monetary policy for an AS-AD model with an outside lag i. Indeed, we find

$$i_t = \pi_t + \bar{r} + h(\pi_t - \pi^*) + b(y_t - \bar{y}),\ h \equiv \frac{1}{\alpha_2\gamma},\ b \equiv \frac{1}{\alpha_2}.$$

Notes

1 In the case of a nonzero correlation between a and ε the optimal policy is

$$x_u = \frac{\bar{a}^2 x_c - \rho\sigma_a\sigma_\varepsilon}{\bar{a}^2 + \sigma_a^2}.$$

If $\rho < 0$, the adjustment in x_c is reduced.

2 The adjustment equation can also be written in the equivalent form

$$\dot{x} = \lambda_1(x - x^*),$$

where λ_1 is the adjustment speed.

3 Any higher order system has an equivalent augmented first-order system. Let a second-order system be described by the matrix equation

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + B_0 x_t + B_1 x_{t-1},$$

the augmented first-order system will be

$$\begin{pmatrix} y_t \\ z_t \\ x_t \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & B_1 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ z_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} B_0 \\ 0 \\ I \end{pmatrix} x_t.$$

4 The deviations x_t, y_t are about some desired and constant objectives X^*, Y^*, such that $x_t \equiv X_t - X^*$ and $y_t \equiv Y_t - Y^*$.

5 A scalar system is studied by Turnovsky [48]. The optimization problem is given by

$$\texttt{min}\ \mathbf{E}\left[my_t^2 + nx_t^2\right] \ m, n \geq 0\ \texttt{s.t.}\ y_t = (a + \lambda_t)y_{t-1} + (b + \omega_t)x_t + \varepsilon_t,$$

where λ_t, ω_t are i.i.d. with zero mean, variances $\sigma_\lambda^2,\ \sigma_\omega^2$ and correlation coefficient ρ. The optimal policy is $x_t = ry_{t-1}$,where $r \equiv \frac{-(abs+\sigma_\lambda\sigma_\omega\rho s)}{n+b^2s+\sigma_\omega^2 s}$ and where s is the solution of the quadratic equation

$$(1 - a^2 - \sigma_\lambda^2)(b^2 + \sigma_\omega^2) + (ab + \sigma_\lambda\sigma_\omega)^2 s^2 \\ + n(1 - a^2 - \sigma_\lambda^2) - m(b^2 + \sigma_\omega^2)s - mn = 0.$$

A necessary and sufficient condition to have a unique positive solution is (with $\rho = 0$)

$$\sigma_\lambda^2 < 1 - a^2 + \frac{a^2 b^2}{b_\omega^2},$$

where the variabilities σ_λ^2 and σ_ω^2 vary inversely. Moreover, the stabilization requirement is satisfied for any $a,\ b(b \neq 0)$ and any k such that $-1 < a + bk < 1$.

6 More explicitly, the log-linearized condition for the goods market equilibrium is deduced from the consumption function $C = C(Y - T, r, V, \varepsilon)$, with $0 < C'_Y < 1$, $C'_r < 0$ or $C'_r > 0$,$C'\varepsilon > 0$, the balanced government budget condition $G = T$, the investment function $I = I(Y, r, \varepsilon)$ with $I'_Y > 0$, $I'_r < 0$, $I'_\varepsilon > 0$ and the equilibrium condition $Y = D(Y, G, r, \varepsilon) + G$ with $0 < D'_Y < 1$, $D'_G < 0$, $D'_r < 0$ and $D'_\varepsilon > 0$. Taking the percentage deviation of the linear approximation in term of relative changes, we have

$$\frac{Y - \bar{Y}}{\bar{Y}} = \tilde{m}(1 - C'_Y)\left(\frac{\bar{G}}{\bar{Y}}\right)\left(\frac{G - \bar{G}}{\bar{G}}\right) + \tilde{m}\left(\frac{D'_r}{\bar{Y}}\right)(r - \tilde{r}) + \tilde{m}\left(\frac{\bar{\varepsilon}D'_\varepsilon}{\bar{Y}}\right)\left(\frac{\varepsilon - \bar{\varepsilon}}{\bar{\varepsilon}}\right),$$

where $\tilde{m} \equiv (1 - D'_Y)^{-1}$. Let $y \equiv \ln Y$ and $g \equiv \ln G$, we deduce

$$y_t - \bar{y} = \alpha_1(g_t - \bar{g}) - \alpha_2(r_t - \bar{r}) + v_t,$$

with coefficients

$$\alpha_1 = \tilde{m}(1 - C'_Y)\left(\frac{\bar{G}}{\bar{Y}}\right),\ \alpha_2 = -\tilde{m}\frac{D'_r}{\bar{Y}},\ \texttt{and}\ v = \tilde{m}\frac{\bar{\varepsilon}D'_\varepsilon}{\bar{Y}}(\ln\varepsilon - \ln\bar{\varepsilon}).$$

7 The money market equilibrium is given by

$$\frac{M}{P} = kY^\eta \times e^{-\beta i},\ k > 0,\ \beta > 0,\ \eta > 0.$$

The supply of real money balances equals a real demand money function $L(Y, i)$ of output Y and nominal interest rate i. When money and prices grow at constant rate, we have $M = (1 + \mu)M_{-1}$ and $P = (1 + \pi)P_{-1}$, where M_{-1} and P_{-1} are the values prevailing in the previous period. Taking the natural logarithms of the money market equilibrium and using the linear approximation of the log function, we have $\mu - \pi + \ln L^* = \ln k + \eta y - \beta i$, where L^* is the long term real demand for money. The long term equilibrium is such that $\pi = \mu$, $Y = \bar{Y}$ and $r = r^*$ and we have $L^* = k\bar{Y}^\eta \times \exp -\beta(\bar{r} + \mu)$. Inserting the expression $\ln L^* = \ln k + \eta\bar{y} - \beta(\bar{r} + \mu)$ in the equilibrium condition and rearranging terms, the nominal interest rate's equation of the model is obtained.

8 The aggregate supply curve is obtained by steps. A link between inflation and unemployment is first deduced from both price and wage setting, and labor demand. Thereafter, the expectations-augmented Phillips curve is introduced. The sectoral production is $Y_i = BL_i^{1-\alpha}$ where B is a productivity parameter. The demand curve is

$$Y_i = \left(\frac{P_i}{P}\right)^{-\sigma} \frac{Y}{n},\ \sigma > 1,$$

where the price elasticity σ evaluates the strength of product market competition. A profit-maximizing firm will fix its output level so that marginal revenue equals marginal costs W_i/MPL_i, where the marginal product of labor is $MPL_i = (1-\alpha)BL_i^{-\alpha}$. The price level is described as the following mark-up over its marginal cost

$$P_i = m^p \frac{W_i}{(1-\alpha)BL_i^{-\alpha}},\ \texttt{with}\ m^p \equiv \frac{\sigma}{\sigma - 1} > 1.$$

The derived sectoral labor demand is given by

$$L_i = \left(\frac{Y}{nB}\right)^{\frac{\varepsilon}{\sigma}} \left(\frac{B(1-\alpha)}{m^p}\right)^{\varepsilon} \left(\frac{W_i}{P}\right)^{-\varepsilon},\ \texttt{with}\ \varepsilon \equiv \frac{\sigma}{1 + \alpha(\sigma - 1)}.$$

Suppose that trade union perfectly controls the level of nominal wage rate and maximizes a sectoral utility function like $\Omega(\omega_i) = (w_i - b)\left(L_i(w_i)\right)^{\eta}$. According to the first order condition of this problem, we deduce $w_i = m^w.b$ with $m^w = \eta\varepsilon(\eta\varepsilon - 1)^{-1}$. The union will also control the real wages W_i/P if it has perfect information about prices. Since at the aggregate level $L = nL_i$ and $Y = nBL_i^{1-\alpha}$, we derive the following relation between inflation and unemployment

$$L = n\left(\frac{B(1-\alpha)P}{m^p m^w P^e}\right)^{\frac{1}{\alpha}}.$$

The level of employment in the long-run equilibrium $\bar{L}$ will be deduced with realized expectations $P^e = P$. Introducing the level of unemployment u with $L = (1-u)N$, the ratio of employment to its trend level $L/\bar{L}$ takes the form

$$\frac{1-u}{1-\bar{u}} = \left(\frac{P}{P^e}\right)^{\frac{1}{\alpha}}\ \texttt{with}\ \bar{u} = 1 - \left(\frac{B(1-\alpha)}{m^p m^w b}\right)^{\frac{1}{\alpha}}.$$

Approximating the logarithm of this expression and taking $\pi = P - P_{-1}$ and $\pi^e = P^e - P_{-1}$, we obtain the expectations-augmented Phillips curve

$$\pi = \pi^e + \alpha(\bar{u} - u).$$

Introducing u in the aggregate production function, taking logarithms and using the approximation $\ln(1-u) \approx -u$, we deduce

$$u = \ln N + \frac{\ln n^{\alpha} + \ln B - y}{1-\alpha}\ \texttt{and}\ \bar{u} = \ln N + \frac{\ln n^{\alpha} + \ln B - \bar{y}}{1-\alpha}.$$

The short-run aggregate supply curve SRAS is then

9 Another monetary policy rule assumes that the monetary authorities react to actual levels of output and inflation on the basis of their rational expectations $y^e_{t,t-1}$ and $\pi^e_{t,t-1}$ respectively. In this case, the demand policies cannot influence real output, since the policy parameters do not appear in the final solution for y_t (see Sørensen and Whitta-Jacobsen [44]).

10 For backward-looking expectations the variances of output and inflation are given by

$$\sigma_y^2 = \frac{2\sigma_z^2 + \alpha^2\sigma_z^2}{\alpha^2\gamma^2 + 2\alpha\gamma},\ \sigma_\pi^2 = \frac{\gamma^2\sigma_z^2 + \sigma_z^2}{\alpha^2\gamma^2 + 2\alpha\gamma},\ \texttt{where}\ \alpha \equiv \frac{\alpha_2 h}{1 + \alpha_2 b},\ \sigma_z^2 \equiv \frac{\sigma_v^2}{(1 + \alpha_2 b)^2}.$$

11 Similarly,higher value of h will stabilize the inflation gap but will increase the instability of output at the same time. The partial derivatives are

$$\frac{\partial\sigma_y}{\partial h} = \frac{\alpha_2(1 + \alpha_2 b)\sigma_s}{(1 + \alpha_2(b + \gamma h))^2} > 0\ \texttt{and}\ \frac{\partial\sigma_\pi}{\partial h} = -\frac{\alpha_2\gamma(1 + \alpha_2 b)\sigma_s}{(1 + \alpha_2(b + \gamma h))^2} < 0$$

12 The commonly used centroid method will take the center of mass. It favors the rule with the output of greatest area. The height method takes the value of the biggest contributor.

13 A smooth representation (π-curve) may be obtained using s- and z-curves. The s-curve is defined by $s(x; a, b) = \left\{ 0, \texttt{if}\ x < a,\ \frac{1}{2}(1+\cos\frac{x-b}{b-a}\pi),\ \texttt{if}\ a \leq x \leq b,\ 1\ \texttt{if}\ x > b \right\}$ and by $z(x; b, c) = \left\{ 0, \texttt{if}\ x < b,\ \frac{1}{2}(1+\cos\frac{x-b}{c-b}\pi),\ \texttt{if}\ b \leq x \leq c,\ 1\ \texttt{if}\ x > c \right\}$. The $\pi - curve$ is then and $\pi((x; a, b, c)) = \min\left\{ s(x; a, b), z(x; b, c) \right\}$.

14 See Braae and Rutherford [7] for fuzzy relations in a FLC and their influences to select more appropriate operations.

15 Choosing an appropriate dimension of the rule sets is discussed by Chopra and al.[11]. The compared rules bases dimension 9(for 3 MFs), 25 (5 MFs), 49 (7 MFs), 81 (9 MFs) and 121 (11 MFs).

16 Scaling factors may be used to modify easily the universe of discourse of inputs. We then have the scaled inputs $K_e e(t)$ and $K_r \dot{e}(t)$

17 The use of closed-loop theory in economics is due to Tustin [50].

18 The differential form of the delay is the production lag $\alpha/(D+\alpha)$ where the operator D is the differentiation w.r.t time. The distribution form is

$$Y(t)=\int_{\tau=0}^{\infty} w(\tau)Z(t-\tau)d\tau,$$

given by the weighting function $w(t)\equiv \alpha e^{-\alpha t}$. The response function is $F(t)=1-e^{-\alpha t}$ for the path of Y following a unit step-change in Z.

19 The Ziegler-Nichols method is a formal PID tuning method : the I and D gains are first set to zero. The P gain is then increased until to a critical gain K_c at which the output of the loop starts to oscillate.Let denote by T_c the oscillation period, the gains are set to $.5K_c$ for a P-control, to $.45K_c+1.2K_p/T_c$ for a PI-control, to $.6K_c+2K_p/T_c+K_pT_c/8$ for a PID-control.

20 The autonomous constant component is ignored since Y is measured from a stationary level

21 Sørensen and Whitta-Jacobsen [44] make a distinction between a inside and outside lags. In the inside lag, inside the policy making system, one can identify three types of lags : a recognition lag due to the fact that the state of the economy cannot be observed immediately, a decision since it take time to decide a change in the economic policy, and an implementation lag when new administrative procedures have to be considered. The outside lag evaluates the time period between the decision of a new policy and its maximum impact on the target variables.

Appendix A
Phillips' PID policies in the multiplier-accelerator model

1. Dynamics of the unregulated model

The unregulated model (with $G = 0$ and $u = 1$) is governed by a linear second-order differential equation in Y, deduced from the system (1.19) to (1.22). We have

$$\ddot{Y} + \Big(\alpha(1-c) + \beta - \alpha\beta v\Big)\dot{Y} + \alpha\beta(1-c)Y(t) = -\alpha\beta,$$

when $t > 0$ with the initial conditions $Y(0) = 0,\ \dot{Y}(0) = -\alpha$. Taking the following values for the parameters : $c = \frac{3}{4}$, $v = \frac{3}{5}$, $\alpha = 4$ ($T = \frac{1}{\alpha} = 3$ months)and $\beta = 1$ (time constant of the lag 1 year), the differential equation is

$$5\ddot{Y} - 2\dot{Y} + 5Y(t) = -20,\ t > 0,$$

with initial conditions $Y(0) = 0,\ \dot{Y}(0) = -4$. The solution of the unregulated model is

$$Y(t) = -4 + 2e^{t/5}(2\cos\frac{2\sqrt{6}}{5} - \sqrt{6}\sin\frac{2\sqrt{6}}{5}),\ t > 0,$$

or

$$Y(t) = -4 + 6.32e^{t/5}\cos(0.98t + 0.89),\ t > 0.$$

The graph of $Y(t)$ is plotted in Fig.A.2(a). The phase diagram in Fig.A.1 shows an unstable equilibrium which justifies stabilization policies.

2. Proportional plus integral plus derivative stabilization policies

The stabilization of the model proposed by Phillips [35] consists of three additive policies : the proportional P- stabilization policy, the proportional+integral PI-stabilization policy, the proportional+integral+derivative PID-stabilization policy. Modifications are introduced by adding terms to the consumption equation (1.2). For a P-stabilization, the consumption equation (1.20) will be

$$C(t) = c.Y(t) - u(t) - \frac{\lambda}{D+\lambda}K_pY(t),$$

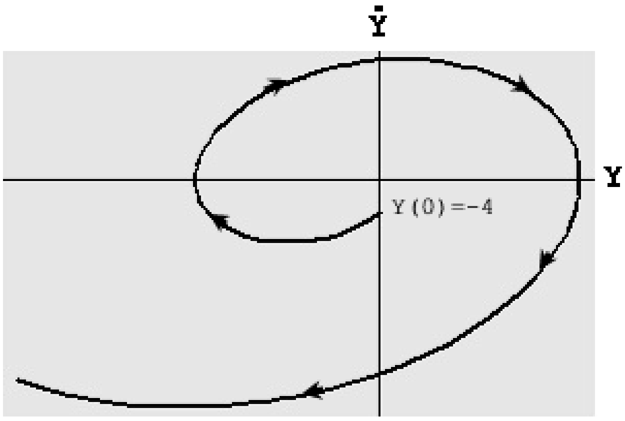

Figure A.1. Phase diagram of the Phillips' model

where K_p denotes the proportional correction factor and λ the speed of response of policy demand to changes in potential policy demand [1]. In the numerical applications, we will retain $\lambda = 2$ (a correction lag with time constant of 6 months). The dynamic equation of the model is a linear third-order differential equation in Y. We have

$$Y^{(3)} + \Big(\alpha(1-c) + \beta + \lambda - \alpha\beta v\Big)\ddot{Y} + \Big(\beta\lambda + (1-c)\alpha(\beta+\lambda) + \alpha\lambda K_p - \alpha\beta\lambda v\Big)\dot{Y} + \alpha\beta\lambda\Big(1 - c + K_p\Big)Y(t) = -\alpha\beta\lambda u(t).$$

Taking $c = \frac{3}{4},\ v = \frac{3}{5},\ \alpha = 4,\ \beta = 1,\ \lambda = 2,\ K_p = 2,\ u = 1$, the differential equation is

$$5Y^{(3)} + 8\ddot{Y} + 81\dot{Y} + 90Y(t) = -40,\ t > 0,$$

with initial conditions $Y(0) = 0,\ \dot{Y}(0) = -4,\ \ddot{Y}(0) = -5.6$. The solution (for $t > 0$) is

$$Y(t) = -.44 - .03e^{-1.15t} - 1.1e^{-.23t}\sin(-3.96t + .44).$$

The graph of the P-controlled $Y(t)$ is plotted in Fig.**??**(b).The system is stable according to the Routh-Hurwitz stability conditions. [2] Moreover, the stability conditions for K_p are $K_p \leq -0.25$ and $K_p \geq 0.35$.

For a PI-stabilization policy, the consumption equation (1.20) will be

$$C(t) = c.Y(t) - u(t) - \frac{\lambda}{D+\lambda}\Big\{K_p Y(t) + K_i \int Y(t)dt\Big\},$$

where K_i denotes the integral correction factor. The dynamic equation of the model is a linear fourth-order differential equation in Y. We have

$$Y^{(4)} + \Big(\alpha(1-c) + \beta + \lambda - \alpha\beta v\Big)Y^{(3)} + \Big(\alpha(1-c)(\beta+\lambda) + \beta\lambda + \alpha\lambda K_p - \alpha\beta\lambda v\Big)\ddot{Y} + \Big(\alpha\beta\lambda(1-c) + \alpha\beta\lambda K_p + \alpha\lambda K_i\Big)\dot{Y}(t) + \alpha\beta\lambda K_i Y(t) = 0.$$

Taking $c = \frac{3}{4},\ v = \frac{3}{5},\ \alpha = 4,\ \beta = 1,\ \lambda = 2,\ K_p = K_i = 2,\ u = 1$, the differential equation is

$$5Y^{(4)} + 8Y^{(3)} + 81\ddot{Y} + 170\dot{Y} + 80Y(t) = 0,\ t > 0,$$

with initial conditions $Y(0) = 0,\ \dot{Y}(0) = -4,\ \ddot{Y}(0) = -5.6,\ Y^{(3)} = 96$. The solution (for $t > 0$) is

$$Y(t) = -.07e^{-1.43t} - .13e^{-.69t} + 1.08e^{.26t}\sin(-4.03t + .19).$$

The graph of the PI-controlled $Y(t)$ is plotted in Fig.**??**(c). The system is unstable, since the Routh-Hurwitz conditions are not all satisfied [3]. Given $K_p = 2$, the stability conditions on K_i are $K_i \in [0, .8987]$.

For a PID-stabilization policy, the consumption equation (1.20) will be

$$C(t) = c.Y(t) - u(t) - \frac{\lambda}{D+\lambda}\left\{K_p Y(t) + K_i \int Y(t)dt + K_d DY(t)\right\},$$

where K_d denotes the derivative correction factor. The dynamic equation of the model is a linear fourth-order differential equation in Y. We have

$$Y^{(4)} + \Big(\alpha(1-c) + \beta + \lambda + \alpha\lambda K_d - \alpha\beta v\Big)Y^{(3)} + \Big((1-c+\lambda K_d - \lambda v)\alpha\beta + (1-c+K_p)\alpha\lambda + \beta\lambda\Big)\ddot{Y} + \Big(\alpha\beta\lambda(1-c+K_p) + \alpha\lambda K_i\Big)\dot{Y} + \alpha\beta\lambda K_i Y(t) = 0.$$

Taking $c = \frac{3}{4},\ v = \frac{3}{5},\ \alpha = 4,\ \beta = 1,\ \lambda = 2,\ K_p = K_i = 2,\ K_d = .55,\ u = 1$, the differential equation is

$$Y^{(4)} + 6Y^{(3)} + 20.6\ddot{Y} + 34\dot{Y} + 16Y(t) = 0,\ t > 0,$$

with initial conditions $Y(0) = 0,\ \dot{Y}(0) = -4,\ \ddot{Y}(0) = 12,\ Y^{(3)} = 2.4$. The solution (for $t > 0$) is

$$Y(t) = -.07e^{-2.16t} - .12e^{-.74t} + 1.40e^{-1.55t}\cos(2.76t + 1.54)$$

The graph of the PID-controlled $Y(t)$ is plotted in Fig.A.2(d). The system is stable, since the Routh-Hurwitz conditions are all satisfied [4]. Given $K_p = K_i = 2$, the stability conditions on K_d are $K_d < -3.92$ and $K_d \geq .07$. The Fig.A.2 illustrates and compares the results The curve without stabilization policy shows the response of the activity Y to the unit initial decrease of demand. The acceleration coefficient ($v = .8$) generates explosive fluctuations [5].The proportional tuning corrects the level of production but not the oscillations. The oscillations grow worse by the integral tuning. The combined PI-stabilization [6] renders the system unstable. The additional derivative stabilization is then introduced and the combined PID-policy stabilize the system.

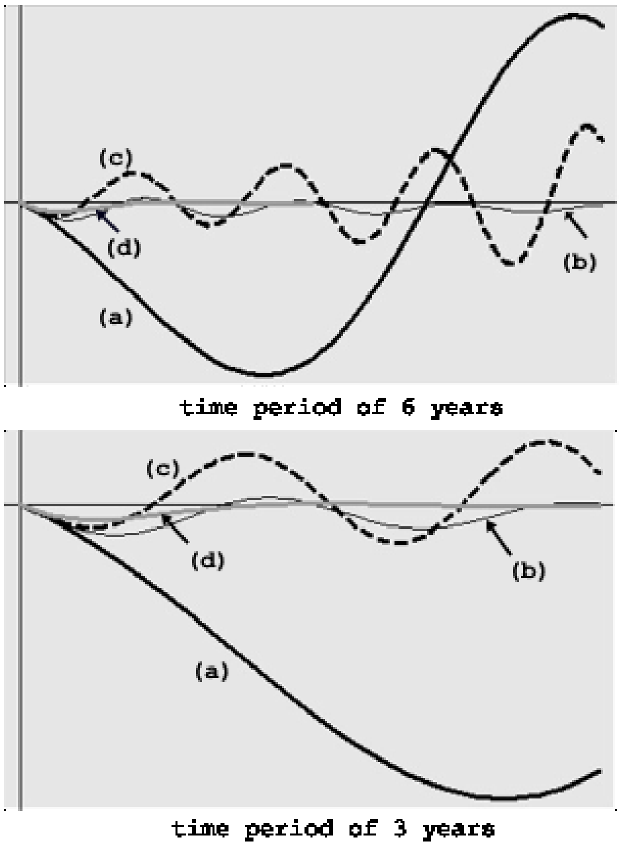

Figure A.2. Stabilization policies over a 3-6 years period : (a)no stabilization policy, (b)P-stabilization policy,(c)PI-stabilization policy, (d)PID-stabilization policy

Notes

1 The time constant of the correction lag is $\frac{1}{\lambda}$years.

2 Let be the polynomial equation with real coefficients

$$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0,\ (a_0 > 0).$$

The Routh-Hurwitz theorem states that necessary and sufficient conditions to have negative real part are given by the conditions that all the leading principal minors of a matrix must be positive. In this case, the 3×3 matrix is

$$\begin{pmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{pmatrix}.$$

We have all the positive leading principal minors : $\Delta_1 = 1$, $\Delta_2 = 7.9$ and $\Delta_3 = 142.5$.

3 We have the leading principal minors :$\Delta_1 = 1$, $\Delta_2 = -8.0$, $\Delta_3 = -274.7$ and $\Delta_3 = -5050.8$.

4 We have the leading principal minors : $\Delta_1 = 1$, $\Delta_2 = 89.6$, $\Delta_3 = 3046.4$ and $\Delta_3 = 39526.4$.

5 Damped oscillations are obtained when the acceleration coefficient lies in the interval $[0, .5]$.

6 The integral correction is rarely used alone.

Appendix B
Control form of a stabilization problem

The optimal stabilization problem with stochastic coefficients is presented first. This initial form, which does not fit to the application of the control theory, is then transformed to a more convenient form. In the control form of the system, the constraint and the objective functions are rewritten. Thereafter, an empirical multiplier-accelerator model is taken for illustrating this procedure.

1. The optimal stabilization problem

Following the Turnovsky's presentation [48], let a system be described by the following matrix equation

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \ldots + A_m Y_{t-m} + B_0 U_t + B_1 U_{t-1} + \ldots + B_n U_{t-n} \qquad \text{(B.1)}$$

The system (B.1) consists in q_1 target variables in instantaneous and delayed vectors *Y*, and q_2 policy instruments in instantaneous and delayed vectors U. The maximum delays are m, and *n* for *Y* and *U* respectively. The squared $q_1 \times q_1$ matrices *A* are associated to the targets, and the $q_1 \times q_2$ matrices *B* are associated to the instruments. All elements of these matrices are subject to stochastic shocks. Suppose that the objective of the policy maker is to stabilize the system close to the long-run equilibrium, the following quadratic objective function may be retained

$$\sum_{t=1}^{\infty}(Y_t - \bar{Y})' M (Y_t - \bar{Y}) + \sum_{t=1}^{\infty}(U_t - \bar{U})' N (U_t - \bar{U}), \qquad \text{(B.2)}$$

where *M* is a strictly positive definite costs matrix associated to the targets and *N* a positive definite matrix associated to the instruments. According to (B.2), the two sets $\bar{Y}$ and u of long-run objectives are required to satisfy

$$\left(I - \sum_{t=1}^{m} A_i\right)\bar{y} = \sum_{t=1}^{n} B_i \bar{u}.$$

Letting the deviations be $Y_t - \bar{Y} = y$ and $U_t - \bar{U} = u$, the optimal stabilization problem is

$$\min_u \sum_{t=1}^{\infty}\Big(y_t' M y_t + \sum_{t=1}^{\infty} u_t' N u_t\Big), \quad \text{s.t.} \tag{B.3}$$

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + \ldots + A_m y_{t-m} + B_0 u_t + B_1 u_{t-1} + \ldots + u_n U_{t-n}.$$

2. The state-space form of the system

The constraint (B.1) is transformed into an equivalent first-order system

$$x_t = A x_t + B v_t$$

Indeed, the constraint may be transformed to the system

$$\begin{cases} y_t & = A_1 y_{t-1} + A_2 y_{t-2} + \ldots + A_m y_{t-m} + B_1 u_{t-1} + \ldots + B_n u_{t-n} + B_0 v_t \\ y_{t-1} & = y_{t-1} \\ \ldots & = \ldots \\ y_{t-(m-1)} & = y_{t-(m-1)} \\ u_t & = u_t \\ \ldots & = \ldots \\ u_{t-(n-1)} & = u_{t-(n-1)} \end{cases}$$

We deduce the $g \times 1$ state vector $\mathtt{x}_t = \begin{pmatrix} y_t \\ y_{t-1} \\ . \\ . \\ . \\ y_{t-(m-1)} \\ u_t \\ . \\ . \\ . \\ u_{t-(n-1)} \end{pmatrix}$ where the y's have q_1 components and the *u*'s, q_2 components. The dimension of the vector is then $g = mq_1 + nq_2$. The control vector is $v_t = u_t$. The associated matrices are

$$\mathtt{A} = \begin{pmatrix} A_1 & A_2 & \ldots & A_m & B_1 & \ldots & B_n \\ I & 0 & \ldots & 0 & 0 & \ldots & 0 \\ . & . & \ldots & . & . & \ldots & . \\ . & . & \ldots & . & . & \ldots & . \\ . & . & \ldots & . & . & \ldots & . \\ 0 & 0 & \ldots & I & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & I & \ldots & 0 \\ . & . & \ldots & . & . & \ldots & . \\ . & . & \ldots & . & . & \ldots & . \\ . & . & \ldots & . & . & \ldots & . \\ 0 & 0 & \ldots & 0 & 0 & \ldots & I \end{pmatrix} \text{ and } \mathtt{B} = \begin{pmatrix} B_0 \\ . \\ . \\ . \\ 0 \\ I \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

The condition for full controllability of the system states that it is possible to move the system from any state to any other. We have

$$\texttt{rank}\Big([B, AB, \ldots, A^{g-1}B]\Big) = g.$$

The objective function (B.2) may be also rewritten as

$$\sum_{t=1}^{\infty}\Big(x_t' M^* xt + \sum_{t=1}^{\infty} v_t' N v_t\Big) - \theta,$$

where θ includes past *y*'s and *u*'s before $t = 1$. Letting $\tilde{M} = M/m$ and $\tilde{N} = N/n$, the block diagonal matrix M^* is defined by

$$\begin{pmatrix} \tilde{M} & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \tilde{M} & \ldots & 0 & 0 & \ldots & 0 \\ . & . & \ldots & . & . & \ldots & . \\ . & . & \ldots & . & . & \ldots & . \\ . & . & \ldots & . & . & \ldots & . \\ 0 & . & \ldots & \tilde{M} & . & \ldots & 0 \\ 0 & . & \ldots & 0 & \tilde{N} & \ldots & 0 \\ . & . & \ldots & . & . & \ldots & . \\ . & . & \ldots & . & . & \ldots & . \\ . & . & \ldots & . & . & \ldots & . \\ 0 & . & \ldots & 0 & 0 & \ldots & \tilde{N} \end{pmatrix}$$

The stabilization problem (B.3)is now transformed to the control form

$$\min_v \sum_{t=1}^{\infty} x_t' M^* x_t + \sum_{t=1}^{\infty} v_t' N v_t,$$

s.t.

$$x_t = Ax_{t-1} + Bv_t.$$

Since the matrices M^* and *N* are strictly positive, the optimal policy exists and is unique.

3. Application to an empirical multiplier-accelerator model

The simple model of an open economy is described by the following equations

$$C_t = 0.8(Y_t - T_t) - 0.05C_{t-1} + 10, \quad \text{(B.4)}$$

$$I_t = 0.2(Y_{t-1} - Y_{t-2}) + 0.2Y_{t-1}, \quad \text{(B.5)}$$

$$T_t = 0.2Y_{t-1} + \theta_t, \quad \text{(B.6)}$$

$$M_t = 0.15Y_t + 1, \quad \text{(B.7)}$$

$$Y_t = C_t + I_t + G_t + X_t - M_t. \quad \text{(B.8)}$$

Equation (B.4) is the consumption function, (B.5) is the investment function, (B.6) is the fiscal equation with parameter θ, (B.7) is the import function and (B.8) determines the income *Y* from the total demand net of imports. Substituting equations (B.5) to (B.7) into (B.6) and (B.8), the system is reduced to

$$C_t - 0.8Y_t = -0.15Y_{t-1} - 0.8\theta_{t-1} - 0.05C_{t-1} + 10, \quad \text{(B.9)}$$

$$-C_t + 1.15Y_t = 0.4Y_{t-1} - 0.2Y_{t-2} + G_{t-1} + X_t - 13, \quad \text{(B.10)}$$

where the control variables G_t and θ_t have been delayed by one period of time. The state variables are C_t, Y_t and Y_{t-1}. Letting $Z_t = Y_{t-1}$, the system (B.9-B.10) may be written

$$\begin{pmatrix} 1.15 & -1 & 0 \\ -0.8 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y_t \\ C_t \\ Z_t \end{pmatrix} = \begin{pmatrix} 0.4 & 0 & -0.2 \\ -0.15 & -0.05 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ C_{t-1} \\ Z_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -0.8 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} G_{t-1} \\ \theta_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_t - 13 \\ 10 \end{pmatrix}.$$

The state matrix equation then is

$$\begin{pmatrix} Y_t \\ C_t \\ Z_t \end{pmatrix} = \begin{pmatrix} 0.7143 & -0.1429 & -0.5714 \\ 0.4214 & -0.1643 & -0.4571 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ C_{t-1} \\ Z_{t-1} \end{pmatrix} + \begin{pmatrix} 2.8571 & -2.2857 \\ 2.2857 & -2.6286 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} G_{t-1} \\ \theta_{t-1} \end{pmatrix} + \begin{pmatrix} 2.8571 & 2.8571 \\ 2.2857 & 3.2857 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_t - 13 \\ 10 \end{pmatrix}.$$

The proper evolution of the system is given by the eigenvalues λ_i's of the matrix A. Since the eigenvalues $\lambda_1 = 0.30 + 0.67i$, $\lambda_2 = 0.30 - 0.67i$ and $\lambda_3 = -0.05$ belong to the unit circle, the system is stable. The imaginary eigenvalues introduce damped oscillations. The system is controllable, since we have $\texttt{rank}\ Q = [B|A.B|A^2.B] = 3$. We have

$$\texttt{rank} \begin{pmatrix} 2.8571 & -2.2857 & 1.7143 & -1.2571 & -0.5265 & 0.4841 \\ 2.2857 & -2.6286 & 0.8286 & -0.5314 & -0.7198 & 0.6024 \\ 0 & 0 & 2.8571 & -2.2857 & 1.7143 & -1.2571 \end{pmatrix} = 3.$$

Appendix C
Spectral analysis of simple macroeconomic models

1. Flexible multiplier-accelerator model without adaptation delay

The equations of the model are

$$\begin{aligned} C_t &= \alpha Y_t,\ 0 < \alpha < 1, & \text{(C.1)} \\ I_t &= \beta Y_t - \delta K_{t-1} + \varepsilon_t,\ \beta, \delta > 0, & \text{(C.2)} \\ I_t &= K_t - K_{t-1}, & \text{(C.3)} \\ Y_t &= C_t + I_t, & \text{(C.4)} \end{aligned}$$

where the endogenous variables are consumption (C), investment (I), the capital stock (K) and output (Y). The stochastic environment is described by the random variable (ε).The equation (C.1)is the consumption function, (C.2) is the investment function with flexible accelerator and random shock, (C.3) deduces investment from the capital stock evolution without depreciation, and (C.4) describes the equilibrium of goods market. The dynamic equation of the output is obtained by eliminating all other endogenous variables. We then have

$$(1 - \alpha - \beta)Y_t - (1 - \alpha - \beta + \delta(1 - \alpha))Y_{t-1} = \varepsilon_t - \varepsilon_{t-1}.$$

We deduce the ARMA(1,1) model

$$Y_t - \frac{1 - \alpha - \beta - +\delta(1 - \alpha)}{1 - \alpha - \beta} Y_t = \varepsilon_t^* - \varepsilon_{t-1}^*,$$

where $\varepsilon_t^* = (1 - \alpha - \beta)^{-1}\varepsilon_t$. Hence, the dynamics of the output Y is $Y_t \rightsquigarrow ARMA(1,1)$. Fig.C.1shows the simulation of the model, taking the following values for parameters : $\alpha = .75$, $\beta = .2$ and $\delta = .05$. The power spectrum is computed by

$$f(\lambda) = 2\sigma_\varepsilon^2 \frac{|\Theta(e^{-j2\pi\lambda}|^2}{|\Phi(e^{-j2\pi\lambda}|^2}.$$

Hence, we find [1]

$$f(\lambda) = 2\sigma_\varepsilon^2 \frac{2(1 - cos2\pi\lambda)}{1 + \phi_1^2 + \phi_2^2 - 2\phi_1(\phi_2 - 1)cos2\pi\lambda - 2\phi_2 cos4\pi\lambda},$$

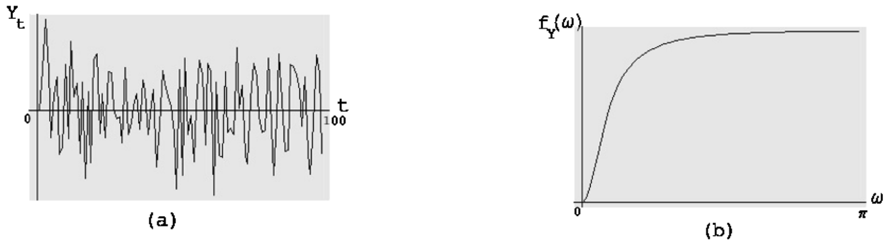

Figure C.1. Simulation results (a) and power spectrum (b)

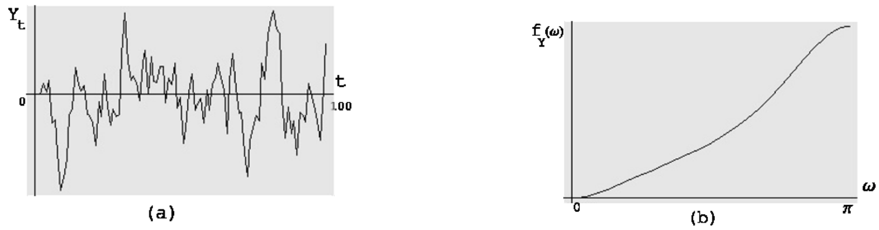

Figure C.2. Simulation results (a) and power spectrum (b)

where

$$\phi_1 \equiv \frac{1-\alpha+\beta-\delta(1-\alpha)}{1-\alpha} \text{ and } \phi_2 \equiv \frac{\beta}{1-\alpha}.$$

Peaks at high frequencies denote short waves.

2. Flexible multiplier-accelerator model with adaptation delay

In such a multiplier-accelerator model, the investment equation (C.2) is replaced by the equation

$$I_t = \beta Y_{t-1} - \delta K_{t-1} + \varepsilon_t, \ \beta, \delta > 0.$$

The final form of the output is the ARMA(2,1) model

$$Y_t - \frac{1-\alpha+\beta-\delta(1-\alpha)}{1-\alpha} Y_{t-1} + \frac{\beta}{1-\alpha} Y_{t-2} = \varepsilon_t^* - \varepsilon_{t-1}^*,$$

where $\varepsilon_t^* \equiv \frac{1}{1-\alpha}\varepsilon_t$. The power spectrum is expressed by

$$f_Y(\omega) = \frac{\sigma_\varepsilon^2}{\pi(1-\alpha)^2} \times \frac{2(1-cos\omega)}{1+\phi_1^2+\phi_2^2+2\phi_1(\phi_2-1)cos\omega - 2\phi_2 cos2\omega}.$$

Fig.C.2 shows the simulation of the model, taking the following values for parameters : $\alpha = .75$, $\beta = .2$ and $\delta = .05$. The presence of a peaks at high frequencies denote short waves.

Notes

1 Indeed, we have

$$\begin{aligned} |\Phi(e^{-j2\pi\lambda})|^2 &= \Phi(e^{-j2\pi\lambda}) \times \overline{\Phi(e^{-j2\pi\lambda})}, \\ &= \left(1 - \phi_1 e^{-j2\pi\lambda} - \phi_2 e^{-j4\pi\lambda}\right) \times \left(1 - \phi_1 e^{j2\pi\lambda} - \phi_2 e^{j4\pi\lambda}\right). \end{aligned}$$

Appendix D
Multiplier-accelerator model with rational expectations

The simple stochastic multiplier-accelerator model for a closed economy with rational expectations is described by three equations. The solution is obtained using both reduced form method and Muth's method of undetermined coefficients.

1. Stochastic multiplier-accelerator model

The equations of the model are

$$C_t = \alpha Y_t + \varepsilon_t,\ 0 < \alpha < 1,\ \mathbf{E}\Big[\varepsilon_t\Big] = 0 \texttt{ for all } t, \tag{D.1}$$

$$I_t = \beta(Y^e_{t,t-1} - Y_{t-1}), \tag{D.2}$$

$$Y_t = C_t + I_t + G_t, \tag{D.3}$$

where $Y^e_{t,t-1} \equiv \mathbf{E}\Big[Y_t | I_{t-1}\Big]$ is the mean value of variable Y at period t, using all information available I_{t-1} at the end of period $t-1$. The equation (D.1) is the consumption equation with shock ε,(D.2) is the investment accelerator and (D.3) states for the equilibrium condition of the economy with exogenous government expenditures G.

2. Solution using the reduced form method

The following reduced form in Y is deduced from the equations (D.1-D.3)

$$Y_t = k_e(Y^e_{t,t-1} - Y_{t-1}) + \frac{G_t}{1-\alpha} + \frac{\varepsilon_t}{1-\alpha}, \tag{D.4}$$

where $k_e \equiv 1/(1-\alpha)$ is the keynesian multiplier. Taking the conditional expectations on both sides of (D.5) so that $\mathbf{E}\Big[Y^e_{t,t-1}\Big] \equiv Y^e_{t,t-1}$, we obtain

$$Y^e_{t,t-1} = -\frac{k_e}{1-k_e} Y_{t-1} + \frac{G_t}{(1-\alpha)(1-k_e)} \tag{D.5}$$

Substituting (D.6) into (D.5) and after rearranging terms, we get the stochastic difference equation for Y

$$Y_t = -\frac{k_e}{1-k_e} Y_{t-1} + \frac{G_t}{(1-\alpha)(1-k_e)} + \frac{\varepsilon_t}{1-\alpha}. \qquad \text{(D.6)}$$

3. Solution using the Muth's method of undetermined coefficients

The Muth's method of undetermined coefficients is based on the Wold's decomposition theorem. According to that theorem, we may write for the endogenous variable Y

$$Y_t = \bar{Y} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i}. \qquad \text{(D.7)}$$

Inserting (D.7) into (D.5), we have

$$\bar{Y} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i} = k_e \left\{ \bar{Y} + \sum_{i=1}^{\infty} \pi_i \varepsilon_{t-i} - \left(\bar{Y} + \sum_{i=0}^{\infty} \pi_i \varepsilon_{t-i-1} \right) \right\} + \frac{G_t}{1-\alpha} + \frac{\varepsilon_t}{1-\alpha}.$$

The only value for $\bar{Y}$ is

$$\frac{\bar{G}}{1-\alpha}.$$

By identification, we find the coefficients

$$\pi_0 \equiv \frac{1}{1-\alpha} \texttt{ and } \pi_i \equiv -\frac{k_e}{1-k_e} \pi_{i-1},\ for\ i \geq 1.$$

Let $h_e \equiv -k_e(1-k_e)^{-1}$, the Koyck transformation $Y_t - hY_{t-1}$ of (D.7) gives the equivalent autoregressive form of order 1

$$Y_t = h_e Y_{t-1} + (1-h_e)\bar{Y} + \pi_0 \varepsilon_t \qquad \text{(D.8)}$$

Replacing the expressions of h_e and $\bar{Y}$ in (D.8), the same stochastic difference equation in Y as in (D.6) is obtained.

References

[1] Allen, R.G.D.(1955). The engineer's approach to economic models, *Economica*, Vol.22, No.86,158–168

[2] Allen, R.G.D.(1973). *Mathematical Economics*, The MacMillan Press LTD, London

[3] Anderson B. D. O. and Moore J. B.(1990),*Optimal Control: Linear Quadratic Methods*, New York:Dover Publications, Inc.

[4] Åström K. J.(1970), *Stochastic Control Theory*, Mineola, N.Y.:Dover Publications, Inc.

[5] Bertsekas, D.P.(1987), *Dynamic Programming : Deterministic and Stochastic Models*, Englewood Cliffs, New Jersey : Prentice-Hall, Inc.

[6] Box, G.E.P. and Jenkins G.M.(1976), *Time series analysis and control*, Oakland, Ca: Holden-Day.

[7] Braae, M. and Rutherford, D.-A. (1978), Fuzzy relations in a control setting, *Kybernetes*, **7**:185–188.

[8] Braae M. and Rutherford, D.-A. (1979), Theoretical and Linguistic Aspects of the fuzzy logic controller, *Automatica*, **15**:553–577.

[9] Brainard, W.C.(1967), Uncertainty and the effectiveness of policy, *The American Economic Review Proceedings*, 57:411–425.

[10] Chatfield, C.(1975), *The analysis of time series: theory and practice*, London : Chapman and Hall.

[11] Chopra, S., Mitra, R. and Kumar,V. (2005), Fuzzy controller : choosing an appropriate and smallest rule set, *International Journal of Computational Cognition*, **3**(4):73–79.

[12] Chow, G.C.(1973), Problems of economic policy from the viewpoint of optimal control, *The American Economic Review*, 63(5):825–837.

[13] Chow, G.C.(1972), How much could be gained by optimal stochastic control policies, *Annals of Economic and Social Measurement*, 1/4:391–406.

[14] Chung B.-M. and Oh J.-H. (1993), Control of dynamic systems using fuzzy learning algorithm, *Fuzzy Sets and Systems*, 59:1-14.

[15] Cominos P. and Nunro N.(2002), PID Controllers : recent tuning methods and design to specification, *IEE Proc.-Control Theory*, 149(1):46–53..

[16] Gabisch, G. and Lorenz, H. W. (1989), *Business Cycle Theory : A survey of methods and concepts*, New York:Springer-Verlag.

[17] Gandolfo, G. (1980), *Economic Dynamics : methods and models*, Amsterdam:North-Holland.

[18] Goodwin, R. M. (1951),The nonlinear accelerator and the persistence of business cycles,*Econometrica*, **19**(1):1–17.

[19] Greene, W.H.(1991), *Econometric Analysis*, New York : MacMillan Publishing Co.

[20] Hall, S.G. and Henry S.G.B.(1988), *Macroeconomic Modelling*, Amsterdam : North-Holland.

[21] Holbrook, R. (1972), Optimal economic policy and the problem of instrument instability, *The American Economic Review*, 62(1):57–65.

[22] Johnson, D. B. (1975), Finding all the elementary circuits of a directed graph, *SIAM J. Comput.*, 4(1):77–84.

[23] Kamien, M.I. and Schwartz, N.L.(1991), *Dynamic Optimization : The Calculus of Variations and Optimal Control in Economics and Management*, 2nd ed, New York : North-Holland.

[24] Keller, A.A., N.L.(2008), *Fuzzy control of macroeconomic models*, ICCAS 2008 "International Conference on Control, Automation and Systems", Prague, Czech Republic, July 25-27, 2008.

[25] Kendrick, D.A.(2002), *Stochastic Control for Economic Models*, Second edition, version 2.00, Austin, Texas: The University of Texas.

[26] T. Kitamoto, M. Saeki, K. Ando (1992), CAD Package for Control System on Mathematica, *IEEE*, 448–451.

[27] Klir, G.J. and Yuan, B.(1995), *Fuzzy Sets and Fuzzy Logic : Theory and Applications*, Upper Saddle River, NJ : Prentice-Hall Inc.

[28] B. Kosko B. (1992),"Neural networks and fuzzy systems : a dynamical systems approach to machine intelligence", Englewood Cliffs, NJ:Prentice-Hall Inc.

[29] Lee, C. C. (1990), Fuzzy logic in control systems: fuzzy logic controller-Part I,*IEEE Transactions on Systems, Man, and Cybernetics*, 20(2):404–418.

[30] Leigh J. R. (1980),*Functional Analysis and Linear Control Theory*, New York:Dover Publications, Inc.

[31] Leland, H.E.(1974), Optimal growth in a stochastic environment, *Review of Economic Studies*, 125:75–86.

[32] Ljungqvist, L. and Sargent T.J.(2000),"Recursive Macroeconomic Theory", Mineola, Massachusetts Institute of Technology.

[33] Lutovac, M.D., Tošić D.V. and Evans B.L.(2001), *Filter design for signal using* $MATLAB^{®}$ *and* $MATHEMATICA^{®}$, Upper Saddle River, New Jersey : Prentice Hall.

[34] Passino, K.M. and Yurkovich, St.(1998), *Fuzzy Control*, Menlo Park, California : Addison-Wesley.

[35] Phillips, A.W. (1954), Stabilisation policy in a closed economy, *Economic Journal*, 64(266):290–323.

[36] Phillips, A.W. (1957), Stabilisation policy and the time-forms of lagged responses, *Economic Journal*, 67(266):265–277.

[37] Poole, W.(1957), Optimal choice of monetary policy instruments in a simple stochastic macro-model, *Quarterly Journal of Economics*, 84:197–216.

[38] Pitchford, J.D. and Turnovsky, S.J.(1977), *Applications of Control Theory to Economic Analysis*, Amsterdam : North-Holland.

[39] Preston, A.J.(1974), A dynamic generalization of Tinbergen's theory of policy, *Review of Economic Studies*, 125:65–74.

[40] Rao, M.J.(1987), *Filtering and Control of Macroeconomic Systems*, Amsterdam : North-Holland.

[41] Sage A. S. and White, C. C. ,III, *Optimum system control*, 2nd edition, Englewood Cliffs, N.J. : Prentice Hall, Inc., 1977.

[42] Sargent, T.J.(1979), *Macroeconomic Theory*, New York:Academic Press.

[43] Shone, R. (2002), *Economic Dynamics : Phase Diagrams and their Economic Application*, 2nd ed. Cambridge UK:Cambridge University Press.

[44] Sørensen, P.B. and Whitta-Jacobsen, H J.(2005), *Introducing Advanced Macroeconomics : Growth & Business Cycles*, London:The McGrawHill Companies.

[45] Stachowicz, M.S. and Beall, L.(2003), *MATHEMATICA Fuzzy Logic*, Wolfram Research. http://www.wolfram.com.

[46] Terano, T. Asai K. and Sugeno, M. (1987) *Fuzzy Systems and Its Applications*, New York : Academic Press, Inc.

[47] The MathWorks inc. (2008), User's Guide : "MATLAB 7", "Simulink 7", "Simulink Control Design", "Control System Design", "Fuzzy Logic Toolbox 2".

[48] Turnovsky, S.J.(1974), The stability properties of optimal economic policies, *The American Economic Review*, 64(1):136–148.

[49] Turnovsky, S.J.(1977), *Macoeconomic Analysis and Stabilization Policy*, Cambridge : Cambridge University Press.

[50] Tustin, A.*The Mechanism of Economic Systems*, London:William Heinemann, LTD, 1953.

[51] Wolfram, S. (2003), *The* $MATHEMATICA^{®}$ *Book*, 5th ed, Champaign Illinois, Wolfram Media Inc.,
http://www.wolfram.com.

[52] Ying H.(2000), *Fuzzy Control and Modeling : Analytical Foundations and Applications*, New York : IEEE (The Institute of Electrical and Electronics Engineers) Press.

[53] Zimmermann, H.-J.(2001), *Fuzzy Set Theory and Its Applications*, Boston : Kluwer Academic Publishers.

Contents

List of Figures

List of Tables