Fuzzy Multi-objective Bimatrix Games :

introduction to the computational techniques

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Outline

- Introduction:
  - Fuzzy environment
  - Fuzzy bimatrix games
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  - Equilibrium solution
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  - Problem formulation
  - Equilibrium solution

Introduction

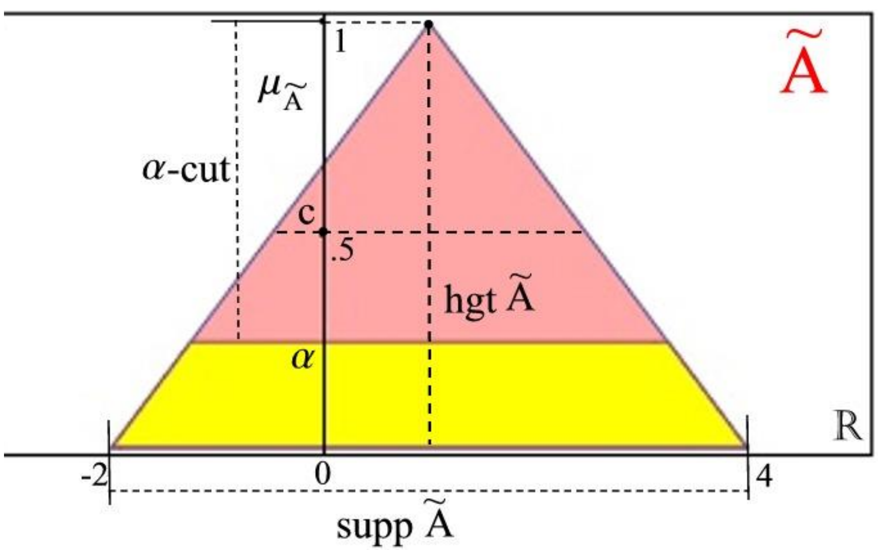
- A. Fuzzy environment
- B. Fuzzy bimatrix games

A. Fuzzy environment (1/2)

☐ **Definition**: Any vague statement $\tilde{S}$ as a fuzzy subset of an universe space X with membership function(MF) $\mu_{\tilde{S}} : X \mapsto [0,1]$.

☐ For any $x \in X$: $\mu_{\tilde{S}}(x) = 1$ means $\tilde{S}$ is "**True**" for x; $\mu_{\tilde{S}}(x) = 0$ means $\tilde{S}$ is "**False**" for x; $\mu_{\tilde{S}}(x) \in (0,1)$ means $\tilde{S}$ is "**possible**" for x with a $\mu_{\tilde{S}}(x)$ degree of possibility.

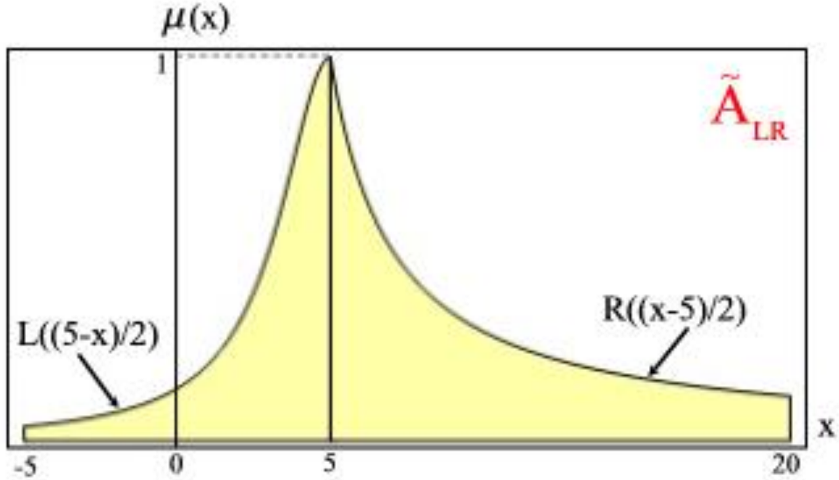
☐ **Definition**: For a piecewise continuous triangular shaped MF: support of $\tilde{A}$ denoted *supp* $\tilde{A} = \{x \in X | \mu_{\tilde{A}}(x) = 0\}$; height of $\tilde{A}$, *hgt* $\tilde{A} = \sup_x \mu_{\tilde{A}}(x)$; crossover points by $c = \{x | \mu_{\tilde{A}}(x) = \frac{1}{2}\}$; alpha-cuts of $\tilde{A}$, s.a. $A_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}$ (see figure).



A. Fuzzy environment (2/2)

☐ **Definition**: Fuzzy number of LR type $\tilde{A}_{LR} = \left(a, \delta_a^-, \delta_a^+\right)$ with reference functions L and R, and positive scalars δ_a^-, δ_a^+ s.t.

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{a-x}{\delta_a^-}), & x \leq a \\ R(\frac{x-a}{\delta_a^+}), & x \geq a. \end{cases}$$



Notation: a, "mean value"; δ_a^-, δ_a^+, left and right spreads.

☐ **Example**: For $L(x) = \frac{1}{1+x^2}$ and $R(x) = \frac{1}{1+2|x|}$ (see figure).

B. Fuzzy bimatrix games (1/3) :

1. Problem formulation (crisp / fuzzy version)

☐ A two-person bimatrix game by

$$G = (S^m, S^n, \mathbf{A}, \mathbf{B})$$

, with strategy spaces S^m, S^n for Players I and II, and $m \times n$ real payoff matrices $\mathbf{A}$, $\mathbf{B}$.

☐ Mixed strategies of Players I and II by $m \times 1$ $\mathbf{x}$ and $n \times 1$ $\mathbf{y}$.

☐ Strategy spaces defined by convex polytopes s.a. $S^m = \{\mathbf{x} \in \mathbb{R}^m_+, \mathbf{x}'\mathbf{e}_m = 1\}$ and $S^n = \{\mathbf{y} \in \mathbb{R}^n_+, \mathbf{y}'\mathbf{e}_n = 1\}$.

☐ Payoffs of Players I and II by two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ with real entries.

☐ Payoff domains of players I and II, by sets $D_1 = \{\mathbf{x'Ay} | \mathbf{x} \in S^m\} \subseteq \mathbb{R}$ and $D_2 = \{\mathbf{x'By} | \mathbf{y} \in S^n\} \subseteq \mathbb{R}$.

☐ Programming problems of the players I and II, represented by $\{\max_\mathbf{x}\ \mathbf{x'Ay}$ subject to $\mathbf{e'}_m\mathbf{x} = 1,\ \mathbf{x} \geq 0\}$ and $\{\max_\mathbf{y}\ \mathbf{x'By}$ subject to $\mathbf{e}'_n\mathbf{y} = 1,\ \mathbf{y} \geq 0\}$, respectively.

☐ A (not completely) fuzzified bimatrix game with fuzzy goals and payoffs by

$$G = (S^m, S^n, \widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}, \tilde{v}, \tilde{p}, \tilde{p}', \tilde{w}, \tilde{q}, \tilde{q}', \lesssim, \gtrsim).$$

Notation: $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}$, fuzzy payoffs $m \times n$ matrices; $\tilde{v}, \tilde{w}$, aspiration levels of Players I and II; $\tilde{p}, \tilde{p}'$, fuzzy tolerance levels for Player I; $\tilde{q}, \tilde{q}'$, fuzzy tolerance levels for Player II; $\lesssim, \gtrsim$, fuzzy inequalities.

☐ A fuzzy goal for Player I by a fuzzy set $\tilde{G}_1$ which membership function (MF) $\mu_1 : D_1 \mapsto [0, 1]$. Player II's fuzzy goal, similarly defined.

☐ LR-representation of fuzzy $\widetilde{\mathbf{A}}$ with entries s.a. $\tilde{a}_{ij} = (a_{ij}, \delta^-_{a_{ij}}, \delta^+_{a_{ij}})_{LR}$. Notation : mean value a_{ij}, left and right spreads, $\delta^-_{a_{ij}}$ and $\delta^+_{a_{ij}}$.

B. Fuzzy bimatrix games (2/3) :
2. Equilibrium solution (crisp version)

☐ **Definition.** Nash equilibrium point $(\mathbf{x}^*, \mathbf{y}^*)$ s.t. players' objectives full filled simultaneously.

$$\mathbf{x}'^*\mathbf{A}\mathbf{y}^* = \max_{\mathbf{x}}\{\mathbf{x}'\mathbf{A}\mathbf{y}^* |\ \mathbf{x}'\mathbf{e}_m = 1,\ \mathbf{x} \geq 0\}$$
$$\mathbf{x}'^*\mathbf{B}\mathbf{y}^* = \max_{\mathbf{y}}\{\mathbf{x}'^*\mathbf{B}\mathbf{y} |\ \mathbf{y}'\mathbf{e}_n = 1,\ \mathbf{y} \geq 0\}$$

Value of the game at the point $(\mathbf{x}'^*\mathbf{A}\mathbf{y}^*, \mathbf{x}'^*\mathbf{B}\mathbf{y}^*)$.

☐ Equivalence Theorem: conditions for $(\mathbf{x}^*, \mathbf{y}^*)$ to be an equilibrium point, as solution of the QP problem :

$$\max_{\mathbf{x},\mathbf{y},p,q} \quad \mathbf{x}'(\mathbf{A}+\mathbf{B})\mathbf{y} - p - q$$
$$\text{subject to}$$
$$\mathbf{A}\mathbf{y} \leq p\mathbf{e}_m,$$
$$\mathbf{B}'\mathbf{x} \leq q\mathbf{e}_n,$$
$$\mathbf{x}'\mathbf{e}_m = 1,$$
$$\mathbf{y}'\mathbf{e}_n = 1,$$
$$\mathbf{x} \geq 0,\ \mathbf{y} \geq 0.$$

B. Fuzzy bimatrix games (3/3) :
2. Equilibrium solution (fuzzy version)

☐ **Definition**: (Bellman-Zadeh decision principle), Player I's fuzzy decision as the intersection of the fuzzy goals and expected payoffs, s.a.for Player I:

$$\mu_{a(\mathbf{x},\mathbf{y})} = \min\left\{\mu_{\mathbf{x}\widetilde{\mathbf{A}}\mathbf{y}}(p), \mu_{\tilde{G}_1}(p)\right\}.$$

Player II's fuzzy decision, similarly defined.

☐ **Definition** : degree of attainment of the fuzzy goal as the maximum of the MF $\mu_{a(\mathbf{x},\mathbf{y})}$:

$$d_1(\mathbf{x},\mathbf{y}) = \max_p\left(\min\left\{\mu_{\mathbf{x}\widetilde{\mathbf{A}}\mathbf{y}}(p), \mu_{\tilde{G}_1}(p)\right\}\right).$$

Player II' s degree similarly defined.

☐ According to the Nishizaki and Sakawa's model, each player maximizes the degree of attainment of his goal. Nash equilibrium solution w.r.t. the degree of attainment of the fuzzy goal as a pair of strategies $(\mathbf{x}^*, \mathbf{y}^*)$ if, for all other strategies:

$$d_1(\mathbf{x}^*,\mathbf{y}^*) \geq d_1(\mathbf{x},\mathbf{y}^*) \text{ for all } \mathbf{x} \in S^m,$$
$$d_2(\mathbf{x}^*,\mathbf{y}^*) \geq d_2(\mathbf{x}^*,\mathbf{y}) \text{ for all } \mathbf{y} \in S^n.$$

I – Single objective fuzzy bimatrix games

1. Problem formulation

☐ Equilibrium solution w.r.t. the degree of attainment of the fuzzy goals by the two players (Nishizaki and Sakawa's model).

☐ Player I's programming problem:

$$\max_{\mathbf{x}} \quad d_1(\mathbf{x}, \mathbf{y}^*) = \frac{\mathbf{x}'(\mathbf{A} + \Delta_{\mathbf{A}})\mathbf{y}^* - \underline{a}}{\bar{a} - \underline{a} + \mathbf{x}'\Delta_{\mathbf{A}}\mathbf{y}^*}$$
$$\text{subject to}$$
$$\mathbf{x}'\mathbf{e}_m = 1,$$
$$\mathbf{x} \geq 0.$$

☐ Player II's programming problem similarly defined.

2. Equilibrium solution

☐ Applying Kuhn-Tucker conditions, equilibrium point $(\mathbf{x}^*, \mathbf{y}^*)$, as solution of a non linear programming problem.

II- Multi-objective fuzzy bimatrix games (1/3)

1. Problem formulation

☐ A multi-objective bimatrix game with uncertain objectives and payoffs.

☐ List of r payoff matrices for Player I by $\widetilde{\mathbf{A}}^k = (\tilde{a}_{ij}^k)_{m\times n},\ k \in \mathbb{N}_r$. List of s payoff matrices for Player II by $\widetilde{\mathbf{B}}^l = (\tilde{b}_{ij}^l)_{m\times n},\ l \in \mathbb{N}_s$.

☐ For triangular fuzzy payoffs, the LR-representation of entries: $\tilde{a}_{ij}^k = (a_{ij}^k, \delta_{a_{ij}}^{k-}, \delta_{a_{ij}}^{k+})_{LR}$ and $\tilde{b}_{ij}^l = (b_{ij}^l, \delta_{b_{ij}}^{l-}, \delta_{b_{ij}}^{l+})_{LR}$.

☐ **Definition** : For any pair of mixed strategies $(\mathbf{x}, \mathbf{y})$, Player I's kth fuzzy expected payoff defined by

$$\mathbf{x}'\widetilde{\mathbf{A}}^k\mathbf{y} = \left(\mathbf{x}'\mathbf{A}^k\mathbf{y}, \mathbf{x}'\Delta_{\mathbf{A}}^{k-}\mathbf{y}, \mathbf{x}'\Delta_{\mathbf{A}}^{k+}\mathbf{y}\right)_{LR}$$

and characterized by the MF

$$\mu_{\mathbf{x}'\widetilde{\mathbf{A}}^k\mathbf{y}} : D_1^k \mapsto [0,1].$$

Player II's lth fuzzy expected payoffs and MF of Player II are similarly defined.

II- Multi-objective fuzzy bimatrix games (2/3)

☐ **Definition** : Player I's kth fuzzy goal G_1^k as a fuzzy set characterized by the MF

$$\mu_1^k : D_1^k \mapsto [0,1].$$

Player II's lth fuzzy goal G_2^l as a fuzzy set characterized by a similar MF.

☐ **Definition**: For any pair of strategies $(\mathbf{x},\mathbf{y})$, an attainment state of the fuzzy goal, represented by the intersection of the fuzzy expected payoffs $\mathbf{x}'\widetilde{\mathbf{A}}^k\mathbf{y}$ and the fuzzy goal $\tilde{G}_1^k$ (minimum component method: seefigure), s.a.

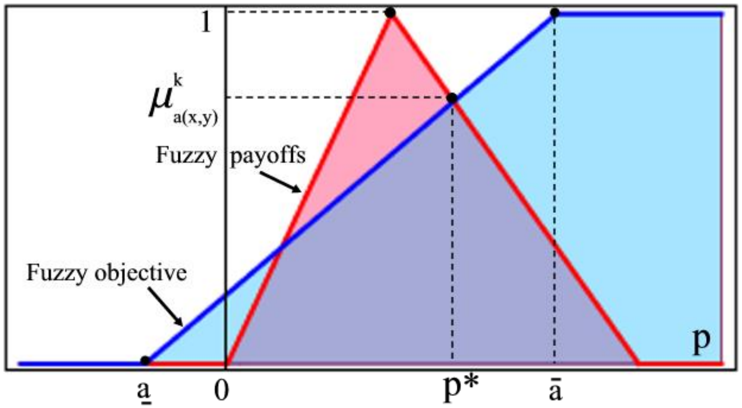
$$\mu^k_{a(\mathbf{x},\mathbf{y})}(p) = \min\left\{\mu^k_{a(\mathbf{x}\widetilde{\mathbf{A}}^k\mathbf{y})}(p), \mu_{\tilde{G}_1^k}(p)\right\},$$

where $p \in D_1^k$, a Player I's payoff.

☐ **Definition** : Degree of attainment of the kth Player I's fuzzy goal as a maximum of the attainment state

$$\hat{\mu}^k_{a(\mathbf{x},\mathbf{y})}(p^*) = \max_p \mu^k_{a(\mathbf{x},\mathbf{y})}(p).$$

Degree of attainment of the fuzzy goal for Player II, similarly defined.



II- Multi-objective fuzzy bimatrix games (3/3)

2. Equilibrium solution

☐ **Definition** : Equilibrium solution w.r.t. the degree of attainment of the aggregated fuzzy goal, a pair of strategies $(\mathbf{x}^*, \mathbf{y}^*)$ s.a. for all other strategies:

$$D^1(\mathbf{x}^*, \mathbf{y}^*) \geq D^1(\mathbf{x}, \mathbf{y}^*), \text{ for all } \mathbf{x} \in S^m$$
$$D^2(\mathbf{x}^*, \mathbf{y}^*) \geq D^2(\mathbf{x}^*, \mathbf{y}), \text{ for all } \mathbf{y} \in S^n.$$

☐ **Definition** : Player I's degree of attainment of the aggregated fuzzy goal, defined by

$$D^1(\mathbf{x}, \mathbf{y}) = \min_{k \in \mathbb{N}_r} \frac{\mathbf{x}'(\mathbf{A}^k + \Delta^k_{\mathbf{A}})\mathbf{y} - \underline{a}^k}{\bar{a}^k - \underline{a}^k + \mathbf{x}'\Delta^k_{\mathbf{A}}\mathbf{y}}.$$

☐ Player I's mathematical programming problem:

$$\max_{\mathbf{x},\sigma} \quad \sigma$$
$$\text{subject to}$$
$$\frac{\mathbf{x}'(\mathbf{A}^k + \Delta^k_{\mathbf{A}})\mathbf{y}^* - \underline{a}^k}{\bar{a}^k - \underline{a}^k + \mathbf{x}'\Delta^k_{\mathbf{A}}\mathbf{y}^*} \geq \sigma,$$
$$\mathbf{x}'\mathbf{e}_m = 1,$$
$$\mathbf{x} \geq 0.$$

Player II's mathematical programming problem, similarly defined.

☐ Applying the Kuhn-Tucker conditions, equilibrium point $(\mathbf{x}^*, \mathbf{y}^*)$ as solution of a nonlinear programming problem.

Thank You for attention !

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suggested questions & complements

1. **Elementary fuzzy logic**

- **Q1**. Fuzzy numbers arithmetics
- **Q2**. Extended addition
- **Q3**. Lattice on fuzzy numbers
- **Q4**. Membership function shapes

2. **Basic principles**

- **Q5**. Min-max problem
- **Q6**. Bellman – Zadeh fuzzy decision principle
- **Q7**. Equivalence Theorems
- **Q8**. Standard fuzzy LP problem

3. **Equilibrium solutions**

- **Q9**. a-Nash equilibrium solution
- **Q10**. Players' beliefs on the Nature 's strategies

4. **Matrix games**

- **Q11**. Matrix games with fuzzy payoffs
- **Q12**. Matrix games with fuzzy goals
- **Q13**. Multi-objective matrix games with fuzzy goals
- **Q14**. Multi-objective matrix games with fuzzy payoffs and fuzzy goals

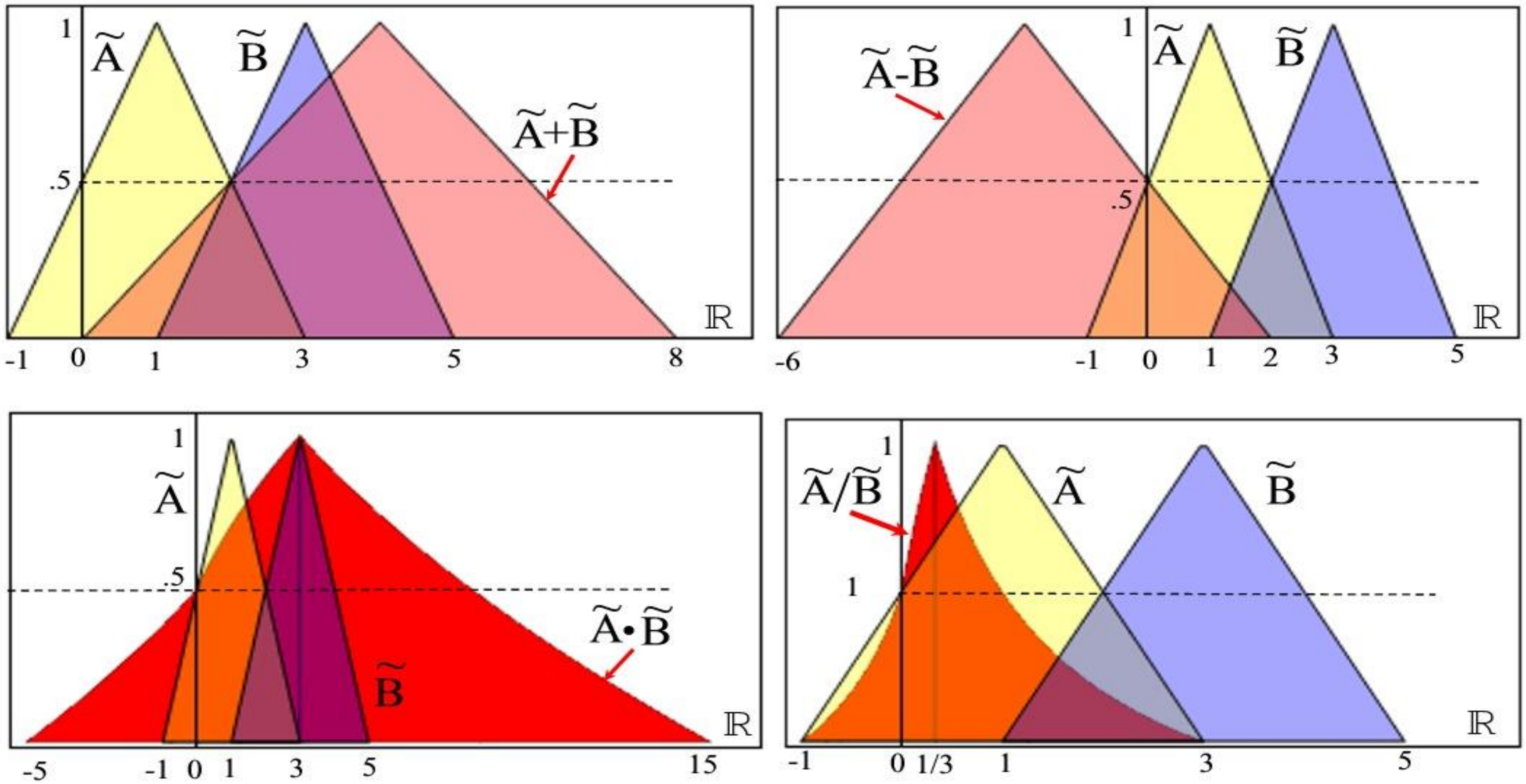
5. **Numerical examples**

- **Q15**. Fuzzy quadratic programming with numerical example
- **Q16**. Numerical example for the single objective FP problem
- **Q17**. Numerical example for the multi-objective FP problem

6. **Literature**

- **Q18**. Application to economics
- **Q19** Computational techniques for fuzzy programming problems
- **Q20**. Main references

Q1. fuzzy numbers arithmetics



Q2. Extented addition

The extension principle will give a method of calculating the MF of the output from the MFs of the input fuzzy quantities. More precisely, let $* : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ with $* \in \{+, -, ., /\}$, a binary operation over real numbers. Then, it can be extended to the operation $\circledast$ over the set $\mathfrak{F}(\mathbb{R})$ of fuzzy quantities.

1. Extension Principle

THEOREM 0.1 *(Extension principle). Denote for $a,\ b \in \mathfrak{F}(\mathbb{R})$ the quantity $c = a * b$ then the MF μ_c is derived from the continuous MFs μ_a and μ_b by the expression*

$$\mu_{a\circledast b}(z) = \sup_{z=x*y} \min\{\mu_a(x), \mu_b(y)\}$$

.

It tells that the possibility that the fuzzy quantity $c = a \circledast b$ achieves $z \in \mathbb{R}$ is as great as the most possible of the real $x,\ y$ such that $z = x * y$, where the $a,\ b$ take the values x, y respectively. For the addition, we also have the ordinary convolution

$$\mu_{a \bigoplus b}(z) = \int_0^z \mu_a(x)\mu_b(z-x)dx$$

2. Example

THEOREM 0.2 *(First Decomposition Theorem). For every $\tilde{A} \in \mathfrak{F}(\mathbb{R})$, we have $\tilde{A} = \bigcup_\alpha \tilde{A}_\alpha = \sup_\alpha$.*

Let the MFs of $\tilde{a}$ and $\tilde{b}$ be

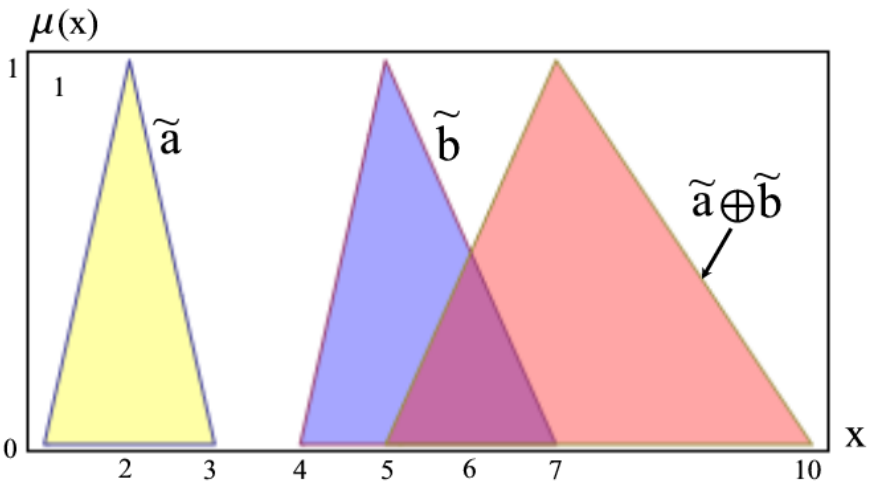
$$\mu_{\tilde{a}}(x) = \begin{cases} x-1,\ x\in[1,2], \\ 3-x,\ x\in[2,3], \\ 0,\ \texttt{otherwise}. \end{cases}$$

and

$$\mu_{\tilde{b}}(x) = \begin{cases} x-4,\ x\in[5,7], \\ (7-x)/2,\ x\in[7,10], \\ 0,\ \texttt{otherwise}. \end{cases}$$

The α-cuts are $\tilde{a}_\alpha = [\alpha+1, 3-\alpha]$ and $\tilde{b}_\alpha = [\alpha+4, 7-2\alpha]$. Then, we have $\tilde{c}_\alpha = (\tilde{a} \bigoplus \tilde{b})_\alpha = \tilde{a}_\alpha + \tilde{b}_\alpha = [2\alpha+5, 10-3\alpha]$. Solving in α, we obtain

$$\mu_{\tilde{a}\bigoplus\tilde{b}}(x) = \begin{cases} (x-5)/2,\ x\in[5,7], \\ (10-x)/3,\ x\in[7,10], \\ 0,\ \texttt{otherwise}. \end{cases}$$

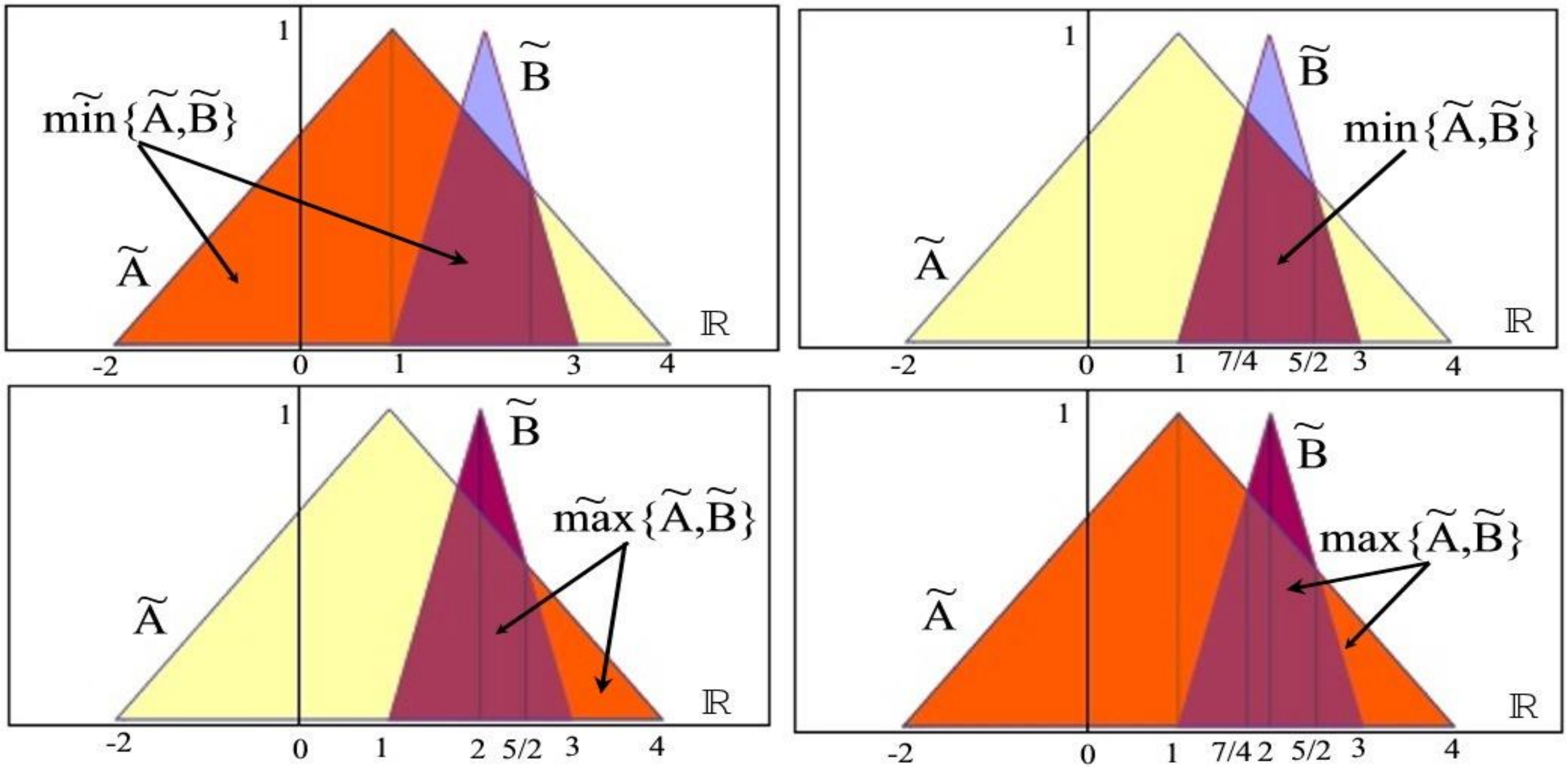


3. Note

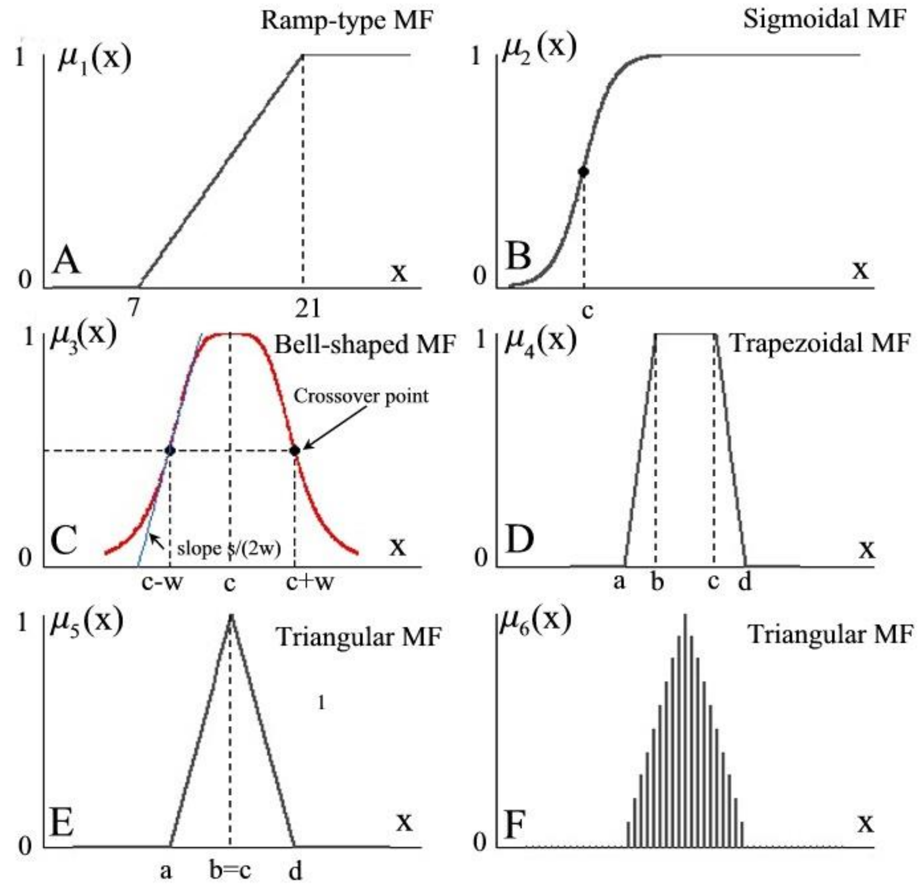
THEOREM 0.3 *(Addition of LR-type FNs). Let $\tilde{a}$ and $\tilde{b}$ two FNs of LR-type $\tilde{a} = (\bar{a}, \delta_a^-, \delta_a^+)_{LR}$ and $\tilde{b} = (\bar{b}, \delta_b^-, \delta_b^+)_{LR}$, then $\tilde{a} \bigoplus \tilde{b} = (\bar{a}+\bar{b}, \delta_a^- + \delta_b^-, \delta_a^+ + \delta_b^+)_{LR}$.*

For $\tilde{a} = (2,1,1)_{LR}$ and $\tilde{b} = (5,1,2)_{LR}$, we simply have $\tilde{a} \bigoplus \tilde{b} = (7,2,3)_{LR}$.

Q3. Lattice on fuzzy numbers



Q4. Membership function shapes



Q5. Min-max problems

A rational optimality criterion is such that the minimizer evaluates his optimal decision against the worst decision, that the maximizer may choose [17]. In a min-max problem, a function to be maximized w.r.t. the maximizer variables is minimized w.r.t. the minimizer variables. A min-max problem with unseparate constraint is defined by

$$\begin{aligned} \min_{\mathbf{x}} \max_{\mathbf{y} \in Y} f(\mathbf{x}, \mathbf{y}) \\ \text{subject to} \\ G(\mathbf{x}, \mathbf{y}) \leq 0, \\ \mathbf{x} \in X = \{\mathbf{x} | g(\mathbf{x}) \leq 0\}. \end{aligned}$$

If the constraints are separated, $G(\mathbf{x}, \mathbf{y}) \leq 0$ does not exist [15]. Let σ be an upper bound of f w.r.t. $\mathbf{y}$ such that $\max_{\mathbf{y} \in Y} f(\mathbf{x}, \mathbf{y}) \leq \sigma$, the min-max problem is transformed into the optimization problem with an infinite number of constraints

$$\begin{aligned} \min_{\mathbf{x}, \sigma} \sigma \\ \text{subject to} \\ f(\mathbf{x}, \mathbf{y}) \leq \sigma, \text{ for all } \mathbf{y} \in Y, \\ \mathbf{x} \in X = \{\mathbf{x} | g(\mathbf{x}) \leq 0.\} \end{aligned}$$

A method to find a solution to this problem is to solve a series of relaxed problems [17]

$$\begin{aligned} \min_{\mathbf{x}, \sigma} \sigma \\ \text{subject to} \\ f(\mathbf{x}, \mathbf{y}^i) \leq \sigma, \text{ for all } \mathbf{y}^i \in Y, \ i \in \mathbb{N}_k \\ \mathbf{x} \in X = \{\mathbf{x} | g(\mathbf{x}) \leq 0.\} \end{aligned}$$

Q6. Bellmann-Zadeh fuzzy decision principle

According to the Bellman-Zadeh symmetry principle, a fuzzy decision set is achieved by using an appropriate aggregation of the fuzzy sets.

Definition 19 *Let X be a set of possible actions, $\{\tilde{G}_j(j \in \mathbb{N}_n\}$ a set of fuzzy objectives, and $\{\tilde{C}_i(i \in \mathbb{N}_m\}$ the decision set is defined by*

$$\tilde{D} = \Big(\bigcap_{j=1}^{n} \tilde{G}_j\Big) \bigcap \Big(\bigcap_{i=1}^{m} \tilde{C}_i\Big),$$

with MF $\mu_{\tilde{D}} : X \mapsto [0,1]$ given by

$$\mu_{\tilde{D}}(x) = \Big(\bigwedge_{j=1}^{n} \mu_{\tilde{G}_j}(x)\Big) \bigwedge \Big(\bigwedge_{i=1}^{m} \mu_{\tilde{C}_i}(x)\Big).$$

The MFs of the aggregate fuzzy goal can be expressed as

$$\mu(x,y) = \min_{k \in \mathbb{N}_r} \Big\{ \mu_k(\mathbf{x}\mathbf{A}^k\mathbf{y}) \Big\}.$$

Hence, we have with linear MFs

$$\mu(x,y) = \min_{k \in \mathbb{N}_r} \Big\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_{ij}^k}{\bar{a}^k - \underline{a}^k} x_i y_j - \frac{\underline{a}^k}{\bar{a}^k - \underline{a}^k} \Big\}.$$

Q7. Equivalence theorems

1. Equivalence theorems

Bimatrix game

Two players I and II have mixed strategies given by the n-dimensional vector $\mathbf{x}$ and the m-dimensional vector $\mathbf{y}$, respectively. The payoffs of players I and II are the $n \times m$ matrices $\mathbf{A}$ and $\mathbf{B}$, respectively. Let $\mathbf{e}_n$ be an n-dimensional vector of ones, $\mathbf{e}_m$ having a dimension m. The objective of player I will be: $\{\max_{\mathbf{x}}\ \mathbf{x}'\mathbf{A}\mathbf{y}$ subject to $\mathbf{e}'_n\mathbf{x} = 1,\ x \geq 0\}$. The objective of player II will then be: $\{\max_{\mathbf{y}}\ \mathbf{x}'\mathbf{B}\mathbf{y}$ subject to $\mathbf{e}'_m\mathbf{y} = 1,\ \mathbf{y} \geq 0\}$.

Equivalence to QP problems

Definition 0.1 *Nash equilibrium. A Nash equilibrium point is is a pair of strategies $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ such that the objectives of the two players are full filled simultaneously. We have*

$$\boldsymbol{x}'^*\boldsymbol{A}\boldsymbol{y}^* = \max_{\boldsymbol{x}}\{\boldsymbol{x}'\boldsymbol{A}\boldsymbol{y}^* | \boldsymbol{e}_n.\boldsymbol{x} = 1, \boldsymbol{x} \geq 0\}$$
$$\boldsymbol{x}'^*\boldsymbol{A}\boldsymbol{y}^* = \max_{\boldsymbol{y}}\{\boldsymbol{x}'^*\boldsymbol{A}\boldsymbol{y} | \boldsymbol{e}_m.\boldsymbol{y} = 1, \boldsymbol{y} \geq 0\}$$

Applying the Kuhn-Tucker necessary and sufficient conditions, we have

Theorem 0.2 *(Equivalence Theorem). A necessary and sufficient condition that $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ be an equilibrium point is it is the solution of the QP problem*

$$\max_{\boldsymbol{x},\boldsymbol{y},\alpha,\beta} \quad \boldsymbol{x}'(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{y} - \alpha - \beta$$
$$\texttt{subject to}$$
$$\boldsymbol{A}\boldsymbol{y} \leq \alpha\boldsymbol{e}_n,$$
$$\boldsymbol{B}'\boldsymbol{x} \leq \beta\boldsymbol{e}_m,$$
$$\boldsymbol{e}'_n\boldsymbol{x} = 1,$$
$$\boldsymbol{e}'_m\boldsymbol{y} = 1,$$
$$\boldsymbol{x} \geq 0,\ \boldsymbol{y} \geq 0,$$

where $\alpha, \beta \in \mathbb{R}$ are the negative of the multipliers associated with the constraints.

PROOF. see O.L. Mangasarian and H. Stone (1964),pp. 350-351.□

Equivalence to LP problems

In zero-sum games we have $\mathbf{B} = -\mathbf{A}$. The QP problems degenerate to two dual problems. We have

$$\max_{\mathbf{x},\gamma} \quad \gamma$$
$$\text{subject to}$$
$$-\mathbf{A}'\mathbf{x} \leq -\gamma\mathbf{e}_n,$$
$$\mathbf{e}'_n\mathbf{x} = 1,$$
$$\mathbf{x} \geq 0.$$

and

$$\min_{\mathbf{x},\alpha} \quad \alpha$$
$$\text{subject to}$$
$$\mathbf{A}\mathbf{y} \leq \alpha\mathbf{e}_m,$$
$$\mathbf{e}'_m\mathbf{y} = 1,$$
$$\mathbf{y} \geq 0.$$

Numerical example with *Mathematica*:

```
Clear[A, B];

A := {{2, -1}, {-1, 1}}; B := {{1, -1}, {-1, 2}}; x := {x1, x2}; y := {y1, y2}; e := {1, 1}

Timing[Maximize[Rationalize[
  {x.(A+B).y - α - β,
   A[[1]].y ≤ α && A[[2]].y ≤ α &&
    Transpose[B][[1]].x ≤ β && Transpose[B][[2]].x ≤ β &&
    e.x == 1 && e.y == 1 &&
    x1 ≥ 0 && x2 ≥ 0 && y1 ≥ 0 && y2 ≥ 0}],
  {x1, x2, y1, y2, α, β}]]
```

$\{61.375, \{0, \{x1 \to \frac{3}{5}, x2 \to \frac{2}{5}, y1 \to \frac{2}{5}, y2 \to \frac{3}{5}, \alpha \to \frac{1}{5}, \beta \to \frac{1}{5}\}\}\}$

```
N[%]
```

{61.375, {0., {x1 → 0.6, x2 → 0.4, y1 → 0.4, y2 → 0.6, α → 0.2, β → 0.2}}}

Q8. Standard fuzzy LP problem

1. LP problem with fuzzy constraint

The fuzzy linear programming problem consists of a crisp objective function and a fuzzy constraint, such that

$$\begin{aligned} \max_{\mathbf{x}} \;\; \mathbf{c}^\top.\mathbf{x} \;\; &(\mathbf{c}, \mathbf{x} \in \mathbb{R}^n) \\ &\text{subject to} \\ \bar{\mathbf{A}}_i.\mathbf{x} \;\precsim \bar{b}_i, \; &(i \in \mathbb{N}_m) \\ \mathbf{x} &\geq 0. \end{aligned}$$

The $\bar{a}_{ij}$, $\bar{b}_i$ are fuzzy numbers of $\mathfrak{F}(\mathbb{R})$ whose values are known with imprecision. The fuzzy inequality $\precsim$ tells that the decision maker (DM) will allow some violation in the accomplishment of the constraint. The membership functions $\mu_i : \mathfrak{F}(\mathbb{R}) \mapsto [0, 1]$, $i \in \mathbb{N}_m$ will measure the adequacy between both sides of the constraint $A_i.x$ and b_i. The FNs $\bar{p}_i$ will express the margins of tolerance for each constraint.

2. Converting the fuzzy constraint

The FLP problem may be written

$$\begin{aligned} \max_{\mathbf{x}} \;\; \mathbf{c}^\top.\mathbf{x} \;\; &(\mathbf{c}, \mathbf{x} \in \mathbb{R}^n) \\ &\text{subject to} \\ \bar{\mathbf{A}}_i.\mathbf{x} \;\leq_{\Re} \bar{b}_i + \bar{p}_i(1-\alpha), \; &(i \in \mathbb{N}_m) \\ \mathbf{x} &\geq 0. \end{aligned}$$

In the constraint, the inequality rule $\leq_{\Re}$ is to be chosen by the DM among several ranking functions (or index) matching each FN into the real line, such that $F : \mathfrak{F}(\mathbb{R}) \mapsto \mathbb{R}$.

3. Solving one auxiliary problem

Let a triangular FN be expressed by $\bar{a} = (a, a^-, a^+)$, where a^-, a^+ are the lower and the upper limit of the support, respectively. Ranking the two fuzzy sides of the inequality may give the following auxiliary parametric LP problem

$$\begin{aligned} \max_{\mathbf{x}} \;\; \mathbf{c}^\top.\mathbf{x} \;\; &(\mathbf{c}, \mathbf{x} \in \mathbb{R}^n) \\ &\text{subject to} \\ (A_i + A_i^- + A_i^+).\mathbf{x} \;\leq (b_i + b_i^- + b_i^+) + (p_i + p_i^- + p_i^+)(1-\alpha), \; &(i \in \mathbb{N}_m) \\ \mathbf{x} &\geq 0. \end{aligned}$$

4. Numerical example

This numerical example is due to Delgado et al (1990). The FLP problem is

$$\begin{aligned} \max_{x_1,x_2} \;\; z &= 5x_1 + 6x_2 \\ &\text{subject to} \\ \bar{3}x_1 + \bar{4}x_2 &\lesssim \widetilde{18}, \\ \bar{2}x_1 + \bar{1}x_2 &\lesssim \tilde{7}, \\ x_1, x_2 &\geq 0. \end{aligned}$$

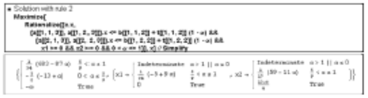
According to the rule that the DM will choose, two different auxiliary problems and solutions are obtained. We have
rule 1 : $\bar{x} \;<_{\Re_1}\; \bar{y} \;\Leftrightarrow\; x \leq y$.

$$\begin{aligned} \max_{x_1,x_2} \;\; z &= 5x_1 + 6x_2 \\ &\text{subject to} \\ 3x_1 + 4x_2 &\leq 18 + 3(1-\alpha), \\ 2x_1 + x_2 &\leq 7 + (1-\alpha), \\ x_1, x_2 &\geq 0, \; \alpha \in (0, 1] \end{aligned}$$

The parametrized solution with rule 1 given by Mathematica is shown hereafter.
rule 2 : $\bar{x} \;<_{\Re_2}\; \bar{y} \;\Leftrightarrow\; \bar{x} \leq \underline{y}$.

$$\begin{aligned} \max_{x_1,x_2} \;\; z &= 5x_1 + 6x_2 \\ &\text{subject to} \\ 4x_1 + 5.5x_2 &\leq 16 + 2.5(1-\alpha), \\ 3x_1 + 2x_2 &\leq 6 + .5(1-\alpha), \\ x_1, x_2 &\geq 0, \; \alpha \in (0, 1] \end{aligned}$$

The parametrized solution with rule 2 given by Mathematica is shown hereafter.



Q9. α-Nash equilibrium solution (1/3)

We consider a bimatrix game $G = (S^m, S^n, \widetilde{\mathbf{A}}, \widetilde{\mathbf{B}})$ with fuzzy payoffs. Suppose that the payoffs are triangular fuzzy numbers (TFNs) of the form $\tilde{a} = (l, m, u)$, where the real numbers l, m and u denote the lower, the middle and the upper value, respectively. The fuzzy payoffs of Player I are represented by $\widetilde{\mathbf{A}} = (\tilde{a}_{ij})_{m\times n}$. The entry $\tilde{a}_{ij}$ denotes the (fuzzy) payoff that Player I receives when the players I and II choose the pure strategy i and j respectively.

Definition 12 *The **α-cut** of a fuzzy number $\tilde{a}$ is defined by $a_\alpha = \{x \in X \,|\mu_{\tilde{a}}(x) \geq \alpha\}$. It can be represented by the closed interval*

$$[\underline{a}_\alpha, \bar{a}_\alpha] = \{\lambda(\bar{a}_\alpha - \underline{a}_\alpha) + \underline{a}_\alpha,\ \lambda \in [0,1]\},$$

where $\underline{a}_\alpha$ and $\bar{a}_\alpha$ denote the real lower and the upper bounds of the elements, respectively.

According to the method for solving classical games under uncertainty, Larbani [10] introduces Nature as a third Player : Nature chooses the payoffs of Players I and II and the two players express their beliefs about the behavior of Nature. The α-cuts of the payoffs of Player I are

$$\begin{aligned} \tilde{\mathbf{A}}_{\alpha_1} &= [\underline{\mathbf{A}}_{\alpha_1}, \bar{\mathbf{A}}_{\alpha_1}], \\ &= \left\{\Lambda(\bar{\mathbf{A}}_{\alpha_1} - \underline{\mathbf{A}}_{\alpha_1}) + \underline{\mathbf{A}}_{\alpha_1}\right\}, \end{aligned}$$

where $\Lambda = (\lambda_{ij})_{m\times n} \in [0,1]$. Similarly, α-cuts of the payoffs of Player II are

$$\begin{aligned} \tilde{\mathbf{B}}_{\alpha_2} &= [\underline{\mathbf{B}}_{\alpha_2}, \bar{\mathbf{B}}_{\alpha_2}], \\ &= \left\{\Pi(\bar{\mathbf{B}}_{\alpha_2} - \underline{\mathbf{B}}_{\alpha_2}) + \underline{\mathbf{B}}_{\alpha_2}\right\}, \end{aligned}$$

where $\Pi = (\pi_{ij})_{m\times n} \in [0,1]$, $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$. Nature will be favorable to Player I (resp. to Player II) if $\lambda_{ij} \in [\frac{1}{2}, 1]$ (resp. $\pi_{ij} \in [\frac{1}{2}, 1]$) and Nature will be unfavorable to those players, otherwise. For the extreme values $\lambda_{ij} = 0$ (resp. $\pi_{ij} = 0$), Player I (resp. Player II) is rather strong pessimistic. For $\lambda_{ij} = 1$ (resp. $\pi_{ij} = 1$) Player I (resp. Player II) is rather strong optimistic. If $\lambda_{ij} = \pi_{ij} = \frac{1}{2}$, Nature has a balanced behavior towards the players [10]. The solution can be found by solving the QP problem

$$\begin{aligned} \max_{\mathbf{x},\mathbf{y},p,q} \quad & \mathbf{x}'(\mathbf{A}(\lambda^0) + \mathbf{B}(\pi^0))\mathbf{y} - p - q \\ & \text{subject to} \\ & \mathbf{A}_j(\lambda^0)\mathbf{y} \leq p\mathbf{e}_m,\ j = 1,2 \\ & \mathbf{B}_i(\pi^0)\mathbf{x} \leq q\mathbf{e}_n,\ i = 1,2 \\ & \mathbf{x}'\mathbf{e}_m = 1, \\ & \mathbf{y}'\mathbf{e}_n = 1, \\ & \mathbf{x} \geq 0,\ \mathbf{y} \geq 0, \end{aligned}$$

Proposition 13 (**α-Nash equilibrium**) *[10] Let T_{ij} and U_{ij} be closed subsets for λ_{ij} and π_{ij} respectively in $[0,1]$. An α-Nash equilibrium $(\boldsymbol{x}^*, \boldsymbol{y}^*, \lambda^0, \pi^0)$ of the game $G = (S^m, S^n, \boldsymbol{A}(\lambda^0), \boldsymbol{B}(\pi^0))$ is such that $\lambda^0 = \min T_{ij}$ and $\pi^0 = \min U_{ij}$.*

Proof: Larbani [10], p. 661. □

Q9. a-Nash equilibrium solution (3/3)

case 1: For Player I, we have $T_{ij} = U_{ij} = [0, \frac{1}{3}]$ and $\lambda^0 = \pi^0 = 0$. The payoff matrices of Players I and II are respectively

$$\mathbf{A}(\lambda^0) = \begin{pmatrix} \frac{355}{2} & 153 \\ 85 & \frac{355}{2} \end{pmatrix}$$

and

$$\mathbf{B}(\pi^0) = \begin{pmatrix} 195 & 130 \\ \frac{235}{2} & 153 \end{pmatrix}.$$

The game has three Nash equilibria [8].The game has two perfect equilibria and one mixed equilibrium. The first perfect Nash equilibrium is

$$(x_1^*, x_2^*) = (0, 1), (y_1^*, y_2^*) = (0, 1)$$

with an expected payoff of 177.5 for Player I and an expected payoff of 153 for Player II. The second perfect Nash equilibrium is

$$(x_1^*, x_2^*) = (1, 0), (y_1^*, y_2^*) = (1, 0)$$

with an expected payoff of 177.5 for Player I and an expected payoff of 195 for Player II. The third mixed Nash equilibrium is

$$(x_1^*, x_2^*) = (.3532, .6468), (y_1^*, y_2^*) = (.2094, .7906)$$

with an expected payoff of 158.1 for Player I and an expected payoff of 144.9 for Player II.

Q9. α-Nash equilibrium solution (2/3)

Larbani [10] introduces the beliefs of the players about the possible values of the payoffs. The α-cuts of the payoffs of Player I are defined by $\mathbf{A}_{\alpha_1} = [\underline{\mathbf{A}}_{\alpha_1}, \bar{\mathbf{A}}_{\alpha_1}]$, where the lower and upper bound matrices $\underline{\mathbf{A}}_{\alpha_1}$ and $\bar{\mathbf{A}}_{\alpha_1}$ denote the lower and upper bound matrices.

$$\underline{\mathbf{A}}_{\alpha_1} = \begin{pmatrix} 175+5\alpha_1 & 150+6\alpha_1 \\ 80+10\alpha_1 & 175+5\alpha_1 \end{pmatrix},$$

$$\bar{\mathbf{A}}_{\alpha_1} = \begin{pmatrix} 190-10\alpha_1 & 158-2\alpha_1 \\ 100-10\alpha_1 & 190-10\alpha_1 \end{pmatrix}.$$

The α-cuts of the payoffs of Player II are similarly defined by $\mathbf{B}_{\alpha_2} = [\underline{\mathbf{B}}_{\alpha_2}, \bar{\mathbf{B}}_{\alpha_2}]$, where $\underline{\mathbf{B}}_{\alpha_2}$ and $\bar{\mathbf{B}}_{\alpha_2}$ are the lower and upper bound matrices.

$$\underline{\mathbf{B}}_{\alpha_2} = \begin{pmatrix} 190+10\alpha_2 & 128+4\alpha_2 \\ 115+5\alpha_2 & 150+6\alpha_2 \end{pmatrix},$$

$$\bar{\mathbf{B}}_{\alpha_2} = \begin{pmatrix} 215-15\alpha_2 & 138-6\alpha_2 \\ 130-10\alpha_2 & 162-6\alpha_2 \end{pmatrix}.$$

If the players choose α-cut levels such as $\alpha_1 = \alpha_2 = \frac{1}{2}$, the α-cut matrices of Player I and Player II are respectively

$$\mathbf{A}_{\frac{1}{2}} = \begin{pmatrix} [175.5, 189] & [150.6, 157.8] \\ [81, 99] & [175.5, 189] \end{pmatrix}$$

and

$$\mathbf{B}_{\frac{1}{2}} = \begin{pmatrix} [191, 213.5] & [128.4, 137.4] \\ [115.5, 129] & [156.6, 161.4] \end{pmatrix}$$

Suppose, as in Larbani [10], that the players may have two types of beliefs. In the first case, the players believe that Nature plays against them. In the second case, Player I believes that Nature is favorable to him, only for the pairs of strategies (1,1) and (2,2), and against him for the other pairs of strategies. Player II still believes that Nature is against him for all pairs of strategies.

Q10. Players beliefs on Nature's strategies

case 3: The profits of Players I and II depend on the strategies that Nature will choose. Nature is favorable to the players in the range $[\frac{1}{2}, 1]$ for λ and μ. The resulting profits in varying λ and μ in the interval $[0, 1]$ are illustrated by the density plots Fig.2. The profits are then increasing when Nature is more favorable (with higher values of λ and μ).

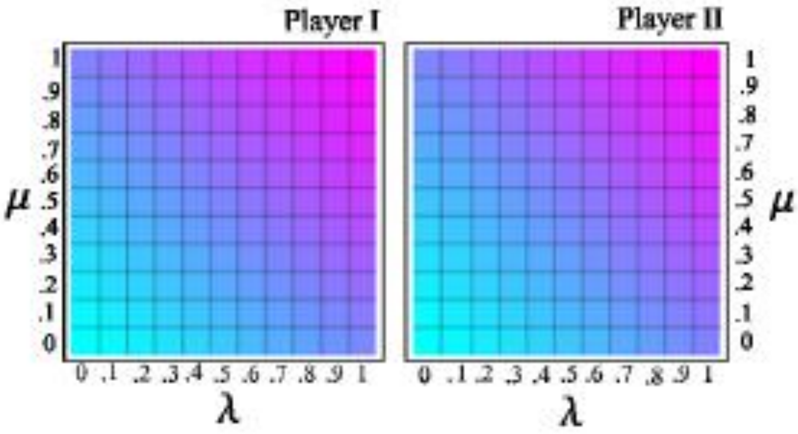


Figure 2: Profits and Nature's strategies

Q11. Matrix Game with fuzzy payoffs (1/2)

1. Problem of the Player I

The fuzzy matrix game problem of the Player I in a fuzzy environment is

$$\max\ v$$
subject to
$$\sum_{i=1}^{m} \tilde{a}_{ij} x_i \gtrsim v,\ (j \in \mathbb{N}_n)$$
$$\sum_{i=1}^{m} x_i = 1,\ x_i \geq 0\ (i \in \mathbb{N}_m).$$

The payoffs of Player I $\tilde{a}_{ij}$ are fuzzy numbers of $\mathfrak{F}(\mathbb{R})$ whose values are known with imprecision. The fuzzy inequality $\precsim$ tells that the decision maker (DM) will allow some violation in the accomplishment of the constraint.

2. Classical changing of the variables

Let change the variables into $u_i = \frac{x_i}{v}\ (i \in \mathbb{N}_m)$. We have $\sum_{i=1}^{m} u_i = \frac{\sum_{i=1}^{m} x_i}{v} = \frac{1}{v}$, then $v = \frac{1}{\sum_{i=1}^{m} u_i}$. The initial problem is transformed to

$$\min \sum_{i=1}^{m} u_i$$
subject to
$$\sum_{i=1}^{m} \tilde{a}_{ij} u_i \gtrsim 1,\ (j \in \mathbb{N}_n)$$
$$u_i \geq 0\ (i \in \mathbb{N}_m).$$

3. Introducing ranking functions

For solving the FLP problem in canonical form, we apply the following procedure : ranking functions are introduced to compare both fuzzy sides of the inequality, and solving a parametric LP problem. The problem of the player I is transformed to

$$\min \sum_{i=1}^{m} u_i$$
subject to
$$\sum_{i=1}^{m} \tilde{a}_{ij} u_i \geq_{\mathfrak{R}} 1 - \tilde{p}_j(1-\alpha),\ (j \in \mathbb{N}_n)$$
$$u_i \geq 0\ (i \in \mathbb{N}_m),\ \alpha \in (0,1].$$

The FNs $\tilde{p}_i$'s are the maximum violation that the player will allow for the constraints.

6. Problem of the Player II

The fuzzy matrix game problem of the Player II in a fuzzy environment is

$$\min\ w$$
subject to
$$\sum_{j=1}^{n} \tilde{a}_{ij} y_j \gtrsim w,\ (i \in \mathbb{N}_m)$$
$$\sum_{j=1}^{n} y_j = 1,\ y_j \geq 0\ (j \in \mathbb{N}_n).$$

The losses of Player II $\tilde{a}_{ij}$ are fuzzy numbers of $\mathfrak{F}(\mathbb{R})$ whose values are known with imprecision. The fuzzy inequality $\lesssim$ tells that the decision maker (DM) will allow some violation in the accomplishment of the constraint.

7. Classical changing of the variables

Let change the variables into $s_j = \frac{y_j}{w}\ (j \in \mathbb{N}_n)$. We have $\sum_{j=1}^{n} s_j = \frac{\sum_{j=1}^{n} y_j}{w} = \frac{1}{w}$, the $w = \frac{1}{\sum_{i=1}^{n} s_j}$. The initial problem is transformed to

$$\max \sum_{j=1}^{n} s_i$$
subject to
$$\sum_{j=1}^{n} \tilde{a}_{ij} s_j \lesssim 1,\ (i \in \mathbb{N}_m)$$
$$s_j \geq 0\ (j \in \mathbb{N}_n).$$

The RHS of the fuzzy inequality is transformed to a crisp number.

8. Introducing ranking functions

For solving the FLP problem in canonical form, we apply the following procedure : ranking functions are introduced to compare both fuzzy sides of the inequality, and solving a parametric LP problem. The problem of the player II is transformed to

$$\max \sum_{j=1}^{m} s_i$$
subject to
$$\sum_{j=1}^{n} \tilde{a}_{ij} s_j \leq_{\mathfrak{R}} 1 + \tilde{q}_i(1-\alpha),\ (i \in \mathbb{N}_m)$$
$$s_j \geq 0\ (j \in \mathbb{N}_n),\ \alpha \in (0,1].$$

The FNs $\tilde{q}_j$'s are the maximum violation that the player will allow for the constraints.

Q11. Matrix game with fuzzy payoffs (2/2)

4. Solving the auxiliary problem

Let the payoffs $\tilde{a}_{ij}$ of Player I be a triangular FNs be expressed by $\tilde{a}_{ij} = (a_{ij}, a_{ij}^-, a_{ij}^+)$, where a_{ij}^-, a_{ij}^+ are the lower and the upper limit of the support, respectively. Ranking the two sides of the inequality may render the following auxiliary parametric LP problem

$$\begin{aligned} \min \sum_{i=1}^{m} u_i \\ \text{subject to} \\ \sum_{i=1}^{m}(a_{ij} + a_{ij}^- + a_{ij}^+)u_i \geq 3 + (p_i + p_i^- + p_i^+)(1-\alpha),\ (j \in \mathbb{N}_n) \\ u_i \geq 0\ (i \in \mathbb{N}_m),\ \alpha \in (0,1]. \end{aligned}$$

5. Numerical example

This numerical example is due to Campos (1989). The fuzzy payoff matrix of Player I is

$$\tilde{A} = \begin{pmatrix} \widetilde{180} & \widetilde{156} \\ \widetilde{90} & \widetilde{180} \end{pmatrix}$$

The FNs are defined by $\widetilde{180} = (180, 175, 190)$, $\widetilde{156} = (156, 150, 158)$, $\widetilde{90} = (90, 80, 100)$. The fuzzy margins are $\tilde{p}_1 = \tilde{p}_2 = (0.10, 0.08, 0.11)$ for the Player I.

FLP problem

$$\begin{aligned} \min\ u_1 + u_2 \\ \text{subject to} \\ \widetilde{180}u_1 + \widetilde{90}u_2 \geq_{\mathcal{R}} 1 - \widetilde{0.10}(1-\alpha) \\ \widetilde{90}u_1 + \widetilde{180}u_2 \geq_{\mathcal{R}} 1 - \widetilde{0.10}(1-\alpha) \\ u_1,\ u_2 \geq 0,\ \alpha \in (0,1]. \end{aligned}$$

Auxiliary problem

$$\begin{aligned} \min\ u_1 + u_2 \\ \text{subject to} \\ 545u_1 + 270u_2 \geq 3 - 0.29(1-\alpha) \\ 464u_1 + 545u_2 \geq 3 - 0.29(1-\alpha) \\ u_1,\ u_2 \geq 0,\ \alpha \in (0,1]. \end{aligned}$$

Solution

We have $x^* = (0.77, 0.23)$ and $v(\alpha) = \frac{482.43}{3-0.29(1-\alpha)}$, $\alpha \in (0,1]$.

9. Solving the auxiliary problem

Let the losses $\tilde{a}_{ij}$ of Player II be a triangular FNs be expressed by $\tilde{a}_{ij} = (a_{ij}, a_{ij}^-, a_{ij}^+)$, where a_{ij}^-, a_{ij}^+ are the lower and the upper limit of the support, respectively. Ranking the two fuzzy sides of the inequality may render the following auxiliary parametric LP problem

$$\begin{aligned} \max \sum_{j=1}^{n} s_i \\ \text{subject to} \\ \sum_{j=1}^{n}(a_{ij} + a_{ij}^- + a_{ij}^+)s_j \leq 3 + (q_i + q_i^- + q_i^+)(1-\alpha),\ (i \in \mathbb{N}_m) \\ s_j \geq 0\ (j \in \mathbb{N}_n),\ \alpha \in (0,1]. \end{aligned}$$

10. Numerical example

The fuzzy losses matrix of Player II is

$$\tilde{A} = \begin{pmatrix} \widetilde{180} & \widetilde{156} \\ \widetilde{90} & \widetilde{180} \end{pmatrix}$$

The FNs are defined by $\widetilde{180} = (180, 175, 190)$, $\widetilde{156} = (156, 150, 158)$, $\widetilde{90} = (90, 80, 100)$. The fuzzy margins are $\tilde{q}_1 = \tilde{q}_2 = (0.15, 0.14, 0.17)$ for the Player II.

FLP problem

$$\begin{aligned} \max\ s_1 + s_2 \\ \text{subject to} \\ \widetilde{180}s_1 + \widetilde{156}s_2 \leq_{\mathcal{R}} 1 - \widetilde{0.15}(1-\alpha) \\ \widetilde{90}s_1 + \widetilde{180}s_2 \leq_{\mathcal{R}} 1 - \widetilde{0.15}(1-\alpha) \\ s_1,\ s_2 \geq 0,\ \alpha \in (0,1]. \end{aligned}$$

Auxiliary problem

$$\begin{aligned} \max\ s_1 + s_2 \\ \text{subject to} \\ 545s_1 + 464s_2 \leq 3 + 0.46(1-\alpha) \\ 270s_1 + 545s_2 \leq 3 + 0.46(1-\alpha) \\ s_1,\ s_2 \geq 0,\ \alpha \in (0,1]. \end{aligned}$$

Solution

We have $y^* = (0.23, 0.77)$ and $w(\alpha) = \frac{482.43}{3+0.49(1-\alpha)}$, $\alpha \in (0,1]$.

Q12. Matrix games with a fuzzy goal (1/2)

Let the single-objective matrix game $G = (S^m, S^n, \mathbf{A})$ with fuzzy goals where S^m and S^n denote the compact convex strategy space of Players, such that $S^m = \{\mathbf{x} \in \mathbb{R}^m_+,\ \mathbf{e}'\mathbf{x} = 1\}$ and $S^n = \{\mathbf{y} \in \mathbb{R}^n_+,\ \mathbf{e}'\mathbf{y} = 1\}$, and where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the payoff matrix of the game with real entries.

1. Fuzzy goal of the Player I

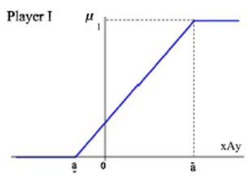
For any pair od strategies (x, y) a membership function $\mu(x, y)$ depends on the expected payoff $\mathbf{xAy}$. Assume that the degree of satisfaction increases linearly, we have

$$\mu(\mathbf{xAy}) = \begin{cases} 1, & \mathbf{xAy} \geq \bar{a} \\ \frac{\mathbf{xAy} - \underline{a}}{\bar{a} - \underline{a}}, & \underline{a} \leq \mathbf{xAy} \leq \bar{a} \\ 0, & \mathbf{xAy} \leq \underline{a}, \end{cases}$$

where $\bar{a}$ and $\underline{a}$ are the best and the worst degree of satisfaction to the Player I, respectively. These extremal values are determined by

$$\begin{aligned} \bar{a} &= \max_x \max_y \mathbf{xAy} = \max_{i \in \mathbb{N}_m} \max_{j \in \mathbb{N}_n} a_{ij} \\ \underline{a} &= \min_x \min_y \mathbf{xAy} = \min_{i \in \mathbb{N}_m} \min_{j \in \mathbb{N}_n} a_{ij} \end{aligned}$$

The membership function of the fuzzy goal is of the shape



2. Player I's maximin solution

Theorem 0.1 *(Maximin solution). For a single-objective two person matrix game with a linearly fuzzy goal function, the Player I's maximin solution w.r.t. a degree of achievement of the fuzzy goal is equal to an optimal solution to the LP problem*

$$\begin{aligned} &\max\ \lambda \\ &\textit{subject to} \\ &\frac{1}{\bar{a} - \underline{a}}\left(\sum_{i=1}^{m} a_{ij} x_i - \underline{a}\right) \geq \lambda, (j \in \mathbb{N}_n) \\ &\boldsymbol{e}'\boldsymbol{x} = 1,\ (\boldsymbol{e}, \boldsymbol{x} \in \mathbb{R}^m), \\ &\boldsymbol{x} \geq 0. \end{aligned}$$

Proof : Nishizaki and Sakawa (2001), p. 39. □

4. Fuzzy goal of the Player II

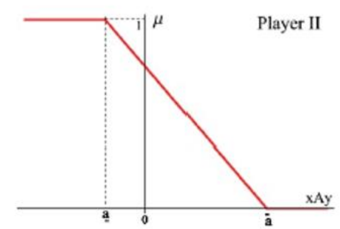
For any pair od strategies (x, y) a membership function $\mu(x, y)$ depends on the expected payoff $\mathbf{xAy}$. Assume that the degree of satisfaction increases linearly, we have

$$\mu(\mathbf{xAy}) = \begin{cases} 1, & \mathbf{xAy} \leq \underline{a} \\ \frac{\bar{a} - \mathbf{xAy}}{\bar{a} - \underline{a}}, & \underline{a} \leq \mathbf{xAy} \leq \bar{a} \\ 0, & \mathbf{xAy} \geq \bar{a}, \end{cases}$$

where $\bar{a}$ and $\underline{a}$ are the worst and the best degree of satisfaction to the Player II, respectively. These extremal values are determined by

$$\begin{aligned} \bar{a} &= \max_x \max_y \mathbf{xAy} = \max_{i \in \mathbb{N}_m} \max_{j \in \mathbb{N}_n} a_{ij} \\ \underline{a} &= \min_x \min_y \mathbf{xAy} = \min_{i \in \mathbb{N}_m} \min_{j \in \mathbb{N}_n} a_{ij} \end{aligned}$$

The membership function of the fuzzy goal is of the shape



Proof : Nishizaki and Sakawa (2001), p. 41. □

5. Player II's minimax solution

Theorem 0.2 *(Minimax solution). For a single-objective two person matrix game with a linearly fuzzy goal function, the Player II's minimax solution w.r.t. a degree of achievement of the fuzzy goal is equal to an optimal solution to the LP problem*

$$\begin{aligned} &\min\ \lambda \\ &\textit{subject to} \\ &\frac{1}{\bar{a} - \underline{a}}\left(\sum_{j=1}^{n} a_{ij} y_i - \underline{a}\right) \leq \lambda + 1, (i \in \mathbb{N}_m) \\ &\boldsymbol{e}'\boldsymbol{y} = 1,\ (\boldsymbol{e}, \boldsymbol{y} \in \mathbb{R}^n), \\ &\boldsymbol{y} \geq 0. \end{aligned}$$

Q12. Matrix game with a fuzzy goal (2/2)

3. Numerical example

In the Cook's example, a 3×3 payoff matrix is given by

$$A = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{pmatrix}$$

We have $\bar{a} = 6$ and $\underline{a} = -2$. Then, we have to solve the LP problem

$$\begin{aligned} &\max_{\mathbf{x},\lambda} \ \lambda \\ &\text{subject to} \\ 2x_1 - x_2 + 2 &\geq 8\lambda, \\ 5x_1 - 2x_2 + 3x_3 + 2 &\geq 8\lambda, \\ x_1 + 6x_2 - x_3 + 2 &\geq 8\lambda, \\ \mathbf{e}.\mathbf{x} = 1,\ \mathbf{x} &\geq 0. \end{aligned}$$

The results of Player I are

Optimality with fuzzy goal (Player I)

```
NMaximize[{λ,
  (B[[1]].x - aw) / (ab - aw) ≥ λ, (B[[2]].x - aw) / (ab - aw) ≥ λ, (B[[3]].x - aw) / (ab - aw) ≥ λ,
  x1 + x2 + x3 == 1, x1 ≥ 0, x2 ≥ 0, x3 ≥ 0}, {x1, x2, x3, λ}]
```

```
{0.453125, {x1 → 0.875, x2 → 0.125, x3 → 0., λ → 0.453125}}
```

6. Numerical example

In the Cook's example, a 3×3 payoff matrix is given by

$$A = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{pmatrix}$$

We have $\bar{a} = 6$ and $\underline{a} = -2$. Then we have to solve the LP problem

$$\begin{aligned} &\min_{\mathbf{y},\lambda} \ \lambda \\ &\text{subject to} \\ 2y_1 + 5y_2 + y_3 + 2 &\leq 8(1+\lambda), \\ -y_1 - 2y_2 + 6x_3 + 2 &\leq 8(1+\lambda), \\ 3y_2 - y_3 + 2 &\leq 8(1+\lambda), \\ \mathbf{e}.\mathbf{y} = 1,\ \mathbf{y} &\geq 0. \end{aligned}$$

The results of Player I are

Optimality with fuzzy goal (Player II)

```
NMinimize[{λ,
  (A[[1]].y - aw) / (ab - aw) ≤ λ + 1, (A[[2]].y - aw) / (ab - aw) ≤ λ + 1, (A[[3]].y - aw) / (ab - aw) ≤ λ + 1,
  y1 + y2 + y3 == 1, y1 ≥ 0, y2 ≥ 0, y3 ≥ 0}, {y1, y2, y3, λ}]
```

```
{-0.546875, {y1 → 0.625, y2 → 0., y3 → 0.375, λ → -0.546875}}
```

Q13. Multi-objective matrix game with fuzzy goals (1/2)

Let the multi-objective matrix game $G = (S^m, S^n, \mathbf{A}_1, \ldots, \mathbf{A}_r)$ with fuzzy goals, where S^m and S^n denote the compact convex strategy spaces of Players, such that $S^m = \{\mathbf{x} \in \mathbb{R}^m_+, \mathbf{e}'\mathbf{x} = 1\}$ and $S^n = \{\mathbf{y} \in \mathbb{R}^n_+, \mathbf{e}'\mathbf{y} = 1\}$, and where the $\mathbf{A}_k'$s $\in \mathbb{R}^{m\times n}$ are the payoff matrices of the game with real entries.

1. Fuzzy goals of the Player I

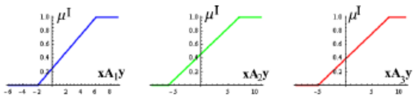
The Player is supposed to have a fuzzy goal for each of the objectives. For any pair od strategies $(\mathbf{x}, \mathbf{y})$ a membership function $\mu_k(\mathbf{x}, \mathbf{y})$ depends on the expected payoff $\mathbf{x}\mathbf{A}_k\mathbf{y}$. Assume that the degree of satisfaction increases linearly, we have

$$\mu(\mathbf{x}\mathbf{A}_k\mathbf{y}) = \begin{cases} 1, & \mathbf{x}\mathbf{A}_k\mathbf{y} \geq \bar{a}_k \\ \frac{\mathbf{x}\mathbf{A}_k\mathbf{y} - \underline{a}_k}{\bar{a}_k - \underline{a}_k}, & \underline{a}_k \leq \mathbf{x}\mathbf{A}_k\mathbf{y} \leq \bar{a}_k \\ 0, & \mathbf{x}\mathbf{A}_k\mathbf{y} \leq \underline{a}_k, \end{cases}$$

where $\bar{a}_k$ and $\underline{a}_k$ are the best and the worst degree of satisfaction to the Player I, respectively. These extremal values are determined by

$$\begin{aligned} \bar{a}_k &= \max_{\mathbf{x}} \max_{\mathbf{y}} \mathbf{x}\mathbf{A}_k\mathbf{y} = \max_{i\in\mathbb{N}_m} \max_{j\in\mathbb{N}_n} a_{k,ij} \\ \underline{a}_k &= \min_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x}\mathbf{A}_k\mathbf{y} = \min_{i\in\mathbb{N}_m} \min_{j\in\mathbb{N}_n} a_{k,ij} \end{aligned}$$

The membership function of the fuzzy goals is of the shape



2. Bellman-Zadeh fuzzy decision rule

According to the Bellman-Zadeh symmetry principle, afuzzy decision set is achieved by using an appropriate aggregation of the fuzzy sets.

Definition 0.1 *Let X be a set of possible actions, $\{\bar{G}_j(j \in \mathbb{N}_n\}$ a set of fuzzy objectives, and $\{\bar{C}_i(j \in \mathbb{N}_m\}$ the decision set is defined by*

$$\bar{D} = \left(\bigcap_{j=1}^{n} \bar{G}_j\right) \bigcap \left(\bigcap_{i=1}^{m} \bar{C}_i\right)$$

with MF $\mu_{\bar{D}} : X \mapsto [0,1]$ given by

$$\mu_{\bar{D}}(x) = \left(\bigwedge_{j=1}^{n} \mu_{\bar{G}_j}(x)\right) \bigwedge \left(\bigwedge_{i=1}^{m} \mu_{\bar{C}_i}(x)\right)$$

.

The MFs of the aggregate fuzzy goal can be expressed as

$$\mu(x, y) = \min_{k\in\mathbb{N}_r} \left\{ \mu_k(\mathbf{x}\mathbf{A}_k\mathbf{y}) \right\}$$

. Hence, we have with linear MFs

$$\mu(x, y) = \min_{k\in\mathbb{N}_r} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_{k,ij}}{\bar{a}_k - \underline{a}_k} x_i y_j - \frac{\underline{a}_k}{\bar{a}_k - \underline{a}_k} \right\}$$

.

5. Fuzzy goals of the Player II

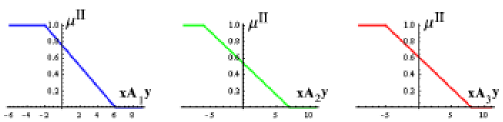
The Player is supposed to have a fuzzy goal for each of the objective. For any pair od strategies $(\mathbf{x}, \mathbf{y})$ a membership function $\mu_k(\mathbf{x}, \mathbf{y})$ depends on the expected payoff $\mathbf{x}\mathbf{A}_k\mathbf{y}$. Assume that the degree of satisfaction decreases linearly, we have

$$\mu(\mathbf{x}\mathbf{A}_k\mathbf{y}) = \begin{cases} 1, & \mathbf{x}\mathbf{A}_k\mathbf{y} \leq \underline{a}_k \\ \frac{\bar{a}_k - \mathbf{x}\mathbf{A}_k\mathbf{y}}{\bar{a}_k - \underline{a}_k}, & \underline{a}_k \leq \mathbf{x}\mathbf{A}_k\mathbf{y} \leq \bar{a}_k \\ 0, & \mathbf{x}\mathbf{A}_k\mathbf{y} \geq \bar{a}_k, \end{cases}$$

where $\bar{a}_k$ and $\underline{a}_k$ are the worst and the best degree of satisfaction to the Player II, respectively. These extremal values are determined by

$$\begin{aligned} \bar{a}_k &= \max_{\mathbf{x}} \max_{\mathbf{y}} \mathbf{x}\mathbf{A}_k\mathbf{y} = \max_{i\in\mathbb{N}_m} \max_{j\in\mathbb{N}_n} a_{k,ij} \\ \underline{a}_k &= \min_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x}\mathbf{A}_k\mathbf{y} = \min_{i\in\mathbb{N}_m} \min_{j\in\mathbb{N}_n} a_{k,ij} \end{aligned}$$

The membership function of the fuzzy goals is of the shape



6. Player II's minimax solution

Theorem 0.3 (*Minimax solution*). *For a multi-objective two person matrix game with a linearly fuzzy goal functions, the Player II's minimax solution w.r.t. a degree of achievement of the fuzzy goals is equal to an optimal solution to the $r+1$ constraints LP problem*

Q13. Multiobjective matrix game with fuzzy goals (2/2)

3. Player I's maximin solution

THEOREM 0.2 *(Maximin solution). For a multi-objective two person matrix game with a linearly fuzzy goal functions, the Player I's maximin solution w.r.t. a degree of achievement of the fuzzy goals is equal to an optimal solution to the $r+1$ constraints LP problem*

$$\max_{x,\lambda} \lambda$$

subject to

$$\frac{1}{\bar{a}_1-\underline{a}_1}\left(\sum_{i=1}^{m} a_{1,ij}x_i-\underline{a}_1\right) \geq \lambda, (j \in \mathbb{N}_n)$$
$$\frac{1}{\bar{a}_2-\underline{a}_2}\left(\sum_{i=1}^{m} a_{2,ij}x_i-\underline{a}_2\right) \geq \lambda, (j \in \mathbb{N}_n)$$
$$\cdots$$
$$\frac{1}{\bar{a}_r-\underline{a}_r}\left(\sum_{i=1}^{m} a_{r,ij}x_i-\underline{a}_r\right) \geq \lambda, (j \in \mathbb{N}_n)$$
$$\boldsymbol{e}'\boldsymbol{x} = 1,\ (\boldsymbol{e},\ \boldsymbol{x} \in \mathbb{R}^m),$$
$$\boldsymbol{x} \geq 0.$$

Proof : Nishizaki and Sakawa (2001), p. 48. □

4. Numerical example

In the Cook's example, three 3×3 payoff matrices are given by

$$A_1 = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & 7 & 2 \\ 0 & -2 & 0 \\ 3 & -1 & -6 \end{pmatrix}, A_3 = \begin{pmatrix} 8 & -2 & 3 \\ -5 & 6 & 0 \\ -3 & 1 & 6 \end{pmatrix}$$

We have $\bar{a}_1 = 6,\ \bar{a}_2 = 7,\ \bar{a}_3 = 8$ and $\underline{a}_1 = -2,\ \underline{a}_2 = -6,\ \underline{a}_3 = -5$. Then, we have to solve the LP problem

$$\max_{\mathbf{x},\lambda} \lambda$$

subject to

$$2x_1 - x_2 + 2 \geq 8\lambda,$$
$$5x_1 - 2x_2 + 3x_3 + 2 \geq 8\lambda,$$
$$x_1 + 6x_2 - x_3 + 2 \geq 8\lambda,$$
$$-3x_1 + 3x_3 + 6 \geq 13\lambda,$$
$$7x_1 - 2x_2 - x_3 + 6 \geq 13\lambda,$$
$$2x_1 - 6x_3 + 6 \geq 13\lambda,$$
$$8x_1 - 5x_2 - 3x_3 + 5 \geq 13\lambda,$$
$$-2x_1 + 6x_2 + x_3 + 5 \geq 13\lambda,$$
$$3x_1 + 6x_3 + 5 \geq 13\lambda,$$
$$\mathbf{e}.\mathbf{x} = 1,\ \mathbf{x} \geq 0.$$

The optimal solution of Player I is $x_1 = 0.59928,\ x_2 = 0.15027,\ x_3 = 0.25045$ with a degree of satisfaction of 38.1 per cent.

Optimality with fuzzy goal (Player I)

[illegible]

$$\min_{y,\lambda} \lambda$$

subject to

$$\frac{1}{\bar{a}_1-\underline{a}_1}\left(\sum_{j=1}^{n} a_{1,ij}y_j-\underline{a}_1\right) \leq \lambda, (i \in \mathbb{N}_m)$$
$$\frac{1}{\bar{a}_2-\underline{a}_2}\left(\sum_{j=1}^{n} a_{2,ij}y_j-\underline{a}_2\right) \leq \lambda, (i \in \mathbb{N}_m)$$
$$\cdots$$
$$\frac{1}{\bar{a}_r-\underline{a}_r}\left(\sum_{j=1}^{n} a_{r,ij}y_j-\underline{a}_r\right) \leq \lambda, (i \in \mathbb{N}_m)$$
$$\boldsymbol{e}'\boldsymbol{y} = 1,\ (\boldsymbol{e},\ \boldsymbol{y} \in \mathbb{R}^n),$$
$$\boldsymbol{y} \geq 0.$$

7. Numerical example

In the Cook's example, three 3×3 payoff matrices is given by

$$A_1 = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & 7 & 2 \\ 0 & -2 & 0 \\ 3 & -1 & -6 \end{pmatrix}, A_3 = \begin{pmatrix} 8 & -2 & 3 \\ -5 & 6 & 0 \\ -3 & 1 & 6 \end{pmatrix}$$

We have $\bar{a}_1 = 6,\ \bar{a}_2 = 7,\ \bar{a}_3 = 8$ and $\underline{a}_1 = -2,\ \underline{a}_2 = -6,\ \underline{a}_3 = -5$. Then, we have to solve the LP problem

$$\min_{\mathbf{y},\lambda} \lambda$$

subject to

$$2y_1 + 5y_2 + y_3 + 2 \leq 8\lambda,$$
$$-y_1 - 2y_2 + 6y_3 + 2 \leq 8\lambda,$$
$$3y_2 - y_3 + 2 \leq 8\lambda,$$
$$-3y_1 + 7y_2 + 2y_3 + 6 \leq 13\lambda,$$
$$-2y_2 + 6 \leq 13\lambda,$$
$$3y_1 - y_2 - 6y_3 + 6 \leq 13\lambda,$$
$$8y_1 - 2y_2 + 3y_3 + 5 \leq 13\lambda,$$
$$-5y_1 + 6y_2 + 5 \leq 13\lambda,$$
$$-3y_1 + y_2 + 6y_3 + 5 \leq 13\lambda,$$
$$\mathbf{e}.\mathbf{y} = 1,\ \mathbf{y} \geq 0.$$

The optimal solution of Player II is $y_1 = 0.38462,\ y_2 = 0.38462,\ y_3 = 0.23077$ with a degree of satisfaction of 61.5 per cent.

Optimality with fuzzy goal (Player II)

[illegible]

Q14. Multiobjective matrix game with fuzzy payoffs and fuzzy goals (1/3)

Let the multi-objective matrix game $G = (S^m, S^n, \bar{\mathbf{A}}_1, \dots, \bar{\mathbf{A}}_r)$ with fuzzy goals and fuzzy constraints, where S^m and S^n denote the compact convex strategy spaces of Players, such that $S^m = \{\mathbf{x} \in \mathbb{R}^m_+, \mathbf{e}'\mathbf{x} = 1\}$ and $S^n = \{\mathbf{y} \in \mathbb{R}^n_+, \mathbf{e}'\mathbf{y} = 1\}$, and where the $\bar{\mathbf{A}}_k$'s $\in \mathbb{R}^{m \times n}$ are the payoff matrices of the game with fuzzy entries $\bar{a}_{ij} \in \mathfrak{F}(\mathbb{R})$.

1. Characterization of the fuzzy expected payoff

Let the fuzzy payoffs for each objective $k \in \mathbb{N}_r$ have the following LR-representation $\bar{a}_{k,ij} = (a_{k,ij}, a^-_{k,ij}, a^+_{k,ij})_{LR}$. The mean value is $a_{k,ij}$ and $a^-_{k,ij}$, $a^+_{k,ij}$ are the left and right spreads, respectively. Using mixed strategies, a fuzzy payoff is extended to a fuzzy expected payoff.

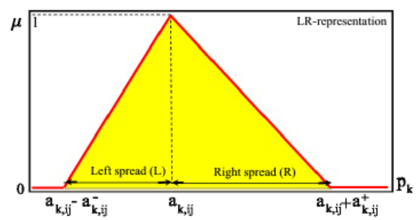
Definition 0.1 *(Fuzzy expected payoff). For any pair of mixed strategies $(\boldsymbol{x}, \mathbf{y})$, the k-th fuzzy expected payoff of Player I is*

$$\boldsymbol{x}\bar{\mathbf{A}}_k\mathbf{y} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{k,ij}x_i y_j, \sum_{i=1}^{m}\sum_{j=1}^{n} a^-_{k,ij}x_i y_j, \sum_{i=1}^{m}\sum_{j=1}^{n} a^-_{k,ij}x_i y_j\right)_{LR}.$$

The MF is such that

$$\mu_{x\bar{\mathbf{A}}y} : D_k \mapsto [0, 1],$$

where D_k is the domain of the k-th payoff for Player I.

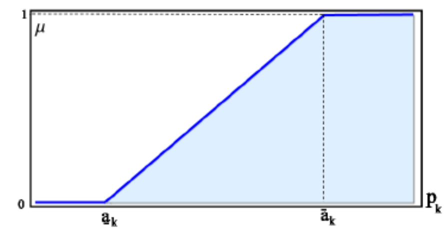


2. Fuzzy goals of the Player I

The Player is supposed to have a fuzzy goal for each of the r objectives. For any pair of mixed strategies $(\mathbf{x}, \mathbf{y})$ a membership function is denoted by $\mu_{\bar{G}_k}$ for the k-th payoff p_k. Assume that the degree of satisfaction increases linearly, we have

$$\mu\left(\bar{G}_k(p_k)\right) = \begin{cases} 1, & p_k > \bar{a}_k \\ \frac{p_k - \underline{a}_k}{\bar{a}_k - \underline{a}_k}, & \underline{a}_k \le p_k \le \bar{a}_k \\ 0, & p_k < \underline{a}_k, \end{cases}$$

where $\bar{a}_k$ and $\underline{a}_k$ are the best and the worst degree of satisfaction to the Player I, respectively. The membership function of the fuzzy goals is of the shape



3. Fuzzy payoffs

The MFs of the payoffs are expressed as

$$\mu(\bar{a}_{k,ij}(p_k) = \begin{cases} \frac{p_k - (a_{k,ij} - a^-_{k,ij})}{a^-_{k,ij}}, & p_k \in [a_{k,ij} - a^-_{k,ij}, a_{k,ij}) \\ \frac{a_{k,ij} + a^+_{k,ij} - p_k}{a^+_{k,ij}}, & p_k \in [a_{k,ij}, a_{k,ij} + a^+_{k,ij}] \\ 0, & p_k \texttt{ not } \in [a_{k,ij} - a^-_{k,ij}, a_{k,ij} - a^+_{k,ij} \end{cases}$$

4. Extension Principle

Let a Cartesian product of universes be $X = X_1 \times \dots \times X_r$ and r fuzzy sets $\bar{A}_1, \dots, \bar{A}_r$ defined on $X_1, X_2, \dots, X_r$ respectively. Let f be a mapping from X to the universe Y, such that $y = f(x_1, x_2, \dots, x_r)$.

Definition 0.2 *The extension principle (Zadeh, 1965,1975) allows to define a fuzzy set $\bar{B}$ on Y trough f from the $\bar{A}_k$' s $(k \in \mathbb{N}_r)$ such that*

$$B = \left\{(y, \mu_{\bar{B}}(y)) | y = f(x_1, x_2, \dots, x_r), (x_1, x_2, \dots, x_r) \in X\right\},$$

where

$$\mu_{\bar{B}}(y) = \begin{cases} \sup_{(x_1,\dots,x_r)\in X} \min\left\{\mu_{\bar{A}_1}(x_1), \dots, \mu_{\bar{A}_r}(x_r)\right\}, & f^{-1} \neq \varnothing \\ 0, & \texttt{otherwise} \end{cases}$$

Example 1: Let $\bar{A}$, $\bar{B}$ be two fuzzy numbers. Using the extension principle the four basic arithmetic operations $* \in \{+, -, ., /\}$ on real numbers are extended to FNs by the expression

$$\mu_{(\bar{A}*\bar{B})}(z) = \sup_{z=x*y} \min\{\mu_{\bar{A}}(x), \mu_{\bar{B}}(y)\}$$

Example 2: For ordering r FNs $\bar{A}_1, \dots, \bar{A}_r$, we consider a priority set P on $\{\bar{A}_1, \dots, \bar{A}_r\}$, such ad $P\left(\bar{A}_k\right)$ is the degree to which $\bar{A}_k$ is ranked as the greatest FN. Using the extension principle, P is defined for each $k \in \mathbb{N}_r$ by the expression

$$P(\bar{A}_k) = \sup \min_{i \in \mathbb{N}_r} \bar{A}_i(u_i),$$

Q14. Multiobjective matrix game with fuzzy payoffs and fuzzy goals (2/3)

where the supremum is taken over all $(u_1, \ldots, u_r) \in \mathbb{R}^r$ such that $u_k \geq u_j$ for all $j \in \mathbb{N}_r$.

Using the extension principle the MF of the k-th expected payoff $\mathbf{x}\mathbf{A}_k\mathbf{y}$ can be represented by the expression

$$\mu_{\mathbf{x}\mathbf{A}_k\mathbf{y}}(p) = \sup_{p=\mathbf{x}\mathbf{P}\mathbf{y}} \min \mu_{\tilde{a}_{k,ij}}(p_{ij}), \; P \in \mathbb{R}^{m \times n}$$

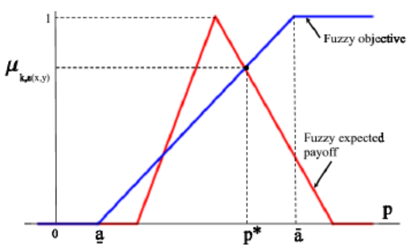
5. Degree of achievement of the aggregated fuzzy goal

Definition 0.3 *(Degree of achievement of a fuzzy goal). For any pair of mixed strategies $(\boldsymbol{x}, \boldsymbol{y})$, let the k-th expected payoff be $\mathbf{x}\tilde{\mathbf{A}}_k\mathbf{y}$ and k-th fuzzy goal be $\tilde{G}_k$. An achievement state of the the fuzzy goal is expressed by the intersection of the fuzzy expected payoff and goal. The MF of the fuzzy set is*

$$\mu_{k,a(x,y)}(p) = \min\left\{\mu_{\mathbf{x}\tilde{\mathbf{A}}_k\mathbf{y}}(p), \mu_{\tilde{G}_k}(p)\right\},$$

where $p \in D_k$ is a payoff of Player I. A degree of achievement of the k-th fuzzy goal is defined as

$$\hat{\mu}_{k,a(\boldsymbol{x},\boldsymbol{y})}(p^*) = \max_p \mu_{k,a(\boldsymbol{x},\boldsymbol{y})}$$



The MF of the aggregated fuzzy goal is $\hat{\mu}_{k,a(x,y)}(p^*)$.

6. Player I's maximin solution

Definition 0.4 *(Maximin solution w.r.t. a degree of achievement of the aggregated fuzzy goal). For any pair of mixed strategies $(\boldsymbol{x}, \boldsymbol{y})$, given a degree of achievement of the aggregated fuzzy goal $\hat{\mu}_{k,a(\boldsymbol{x},\boldsymbol{y})}(p^*)$, the Player I's maximin value w.r.t. a degree of achievement of the aggregated fuzzy goal is*

$$\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \hat{\mu}_{k,a(\boldsymbol{x},\boldsymbol{y})}(p^*).$$

Then we have the expression

$$\max_{x} \min_{y} \min_{k \in \mathbb{N}_r} \max_{p_k} \min\left\{\mu_{\boldsymbol{x}\tilde{A}_k\boldsymbol{y}}(p_k), \mu_{\tilde{G}_k}(p_k)\right\}.$$

Theorem 0.5 *(Player I's maximin solution). For multiobjective two-person matrix games, with linear MFs of the fuzzy goals, and linear shape functions of FNs, the Player I's maximin solution w.r.t. a degree of achievement of the aggregated fuzzy goal is equal to an optimal solution to the nonlinear programming problem*

$$\max_{\boldsymbol{x},\sigma} \sigma$$

subject to

$$\frac{\sum_{i=1}^m \sum_{j=1}^n (a_{k,ij} + a_{k,ij}^+) x_i y_j - \underline{a}_k}{\sum_{i=1}^m \sum_{j=1}^n (a_{k,ij}^+ x_i y_j + \bar{a}_k - \underline{a}_k} \geq \sigma, \; \forall y \in \mathbb{R}^n, \; k \in \mathbb{N}_r$$

$$\boldsymbol{e}'\boldsymbol{x} = 1, \; (\boldsymbol{e}, \boldsymbol{x} \in \mathbb{R}^m),$$

$$\boldsymbol{x} \geq 0.$$

Proof : Nishizaki and Sakawa (2001), p. 65. □

For solving the problem, the algorithm consists in different steps : 1) the use of the relaxation procedure due to Shimizu and Aiyoshi (1980) by taking L points $\mathbf{y}^l, l \in \mathbb{N}_L$ satisfying $\mathbf{e}'\mathbf{y}_l = 1, \; y_l \geq 0$ and obtaining the optimal solution (x^L, σ^L); 2) r minimization problems have then to be solved. The variable transformation by Charnes and Cooper (1962) used for such linear fractional programming problem, induces LP problems.

7. Numerical example

Formulation

A multiobjective two-person matrix game (from Cook, 1976)with fuzzy payoffs and fuzzy goals is analyzed in Nishizaki and Sakawa (2001, p.70). Each Player has three pure strategies. The LR-representation of the fuzzy payoffs is

$$\tilde{\mathbf{A}}_1 = \begin{pmatrix} (2,.2,.2) & (5.5,.5) & (1,.8,.8) \\ (-1,.8,.8) & (-2,.4,.4) & (6,.1,.1) \\ (0,.1,.1) & (3,.5,.5) & (-1,.8,.8) \end{pmatrix},$$

$$\tilde{\mathbf{A}}_2 = \begin{pmatrix} (-3,.8,.8) & (7.3,.3) & (2,.4,.4) \\ (0,.5,.5) & (-2,.2,.2) & (0,.7,.7) \\ (3,.4,.4) & (-1,.8,.8) & (-6,.5,.5) \end{pmatrix},$$

$$\tilde{\mathbf{A}}_3 = \begin{pmatrix} (8,.1,.1) & (-2,.5,.5) & (3,.7,.7) \\ (-5,.5,.5) & (6,.4,.4) & (0,.6,.6) \\ (-3,.8,.8) & (1,.6,.6) & (6,.1,.1) \end{pmatrix}.$$

The fuzzy goals $\tilde{G}_1$, $\tilde{G}_2$, $\tilde{G}_3$ for the three objectives of Player I are represented by linear MFs

$$\mu_{\tilde{G}_1}(p_1) = \begin{cases} 1, \; p_1 > 6.5 \\ (p_1+1)/7.5, \; -1 \leq p_1 \leq 6.5 \\ 0, \; p_1 \leq -1 \end{cases}$$

$$\mu_{\tilde{G}_1}(p_2) = \begin{cases} 1, \; p_2 > 5.5 \\ (p_2+2)/7.5, \; -1 \leq p_2 \leq 6.5 \\ 0, \; p_2 \leq -2 \end{cases}$$

$$\mu_{\tilde{G}_1}(p_3) = \begin{cases} 1, \; p_3 > 5.8 \\ (p_3+1)/6.8, \; -1 \leq p_3 \leq 5.8 \\ 0, \; p_3 \leq -1 \end{cases}$$

Q14. Multiobjective matrix game with fuzzy payoffs and fuzzy goals (3/3)

Shimizu and Aiyoshi 's iterative method

The solution for Player I is obtained after three iterations. We have

$$x_1 = .4434,\ x_2 = .3178,\ x_3 = .2388.$$

Sakawa's direct method : Player I

Using properties of some constraints, the initial nonlinear programming problem for Player I, Sakawa (1983) retains the following equivalent programming problem

$$\max_{\mathbf{x},\sigma}\ \sigma$$

subject to

$$\frac{\sum_{i=1}^{m}(a_{k,ij} + a^{+}_{k,ij})x_i - \underline{a}_k}{\sum_{i=1}^{m} a^{+}_{k,ij}x_i + \bar{a}_k - \underline{a}_k} \geq \sigma,\ j \in \mathbb{N}_n,\ k \in \mathbb{N}_r$$

$$\mathbf{e}'\mathbf{x} = 1,\ (\mathbf{e},\ \mathbf{x} \in \mathbb{R}^m)$$

$$\mathbf{x} \geq 0.$$

The results for Player I are obtained with a degree of satisfaction of 24.6 per cent.

```
NMaximize[{σ,
  ((B1[[1]] + F1[[1]]).x - aw1) / (F1[[1]].x + ab1 - aw1) ≥ σ,
  ((B1[[2]] + F1[[2]]).x - aw1) / (F1[[2]].x + ab1 - aw1) ≥ σ,
  ((B1[[3]] + F1[[3]]).x - aw1) / (F1[[3]].x + ab1 - aw1) ≥ σ,
  ((B2[[1]] + F2[[1]]).x - aw2) / (F2[[1]].x + ab2 - aw2) ≥ σ,
  ((B2[[2]] + F2[[2]]).x - aw2) / (F2[[2]].x + ab2 - aw2) ≥ σ,
  ((B2[[3]] + F2[[3]]).x - aw2) / (F2[[3]].x + ab2 - aw2) ≥ σ,
  ((B3[[1]] + F3[[1]]).x - aw3) / (F3[[1]].x + ab3 - aw3) ≥ σ,
  ((B3[[2]] + F3[[2]]).x - aw3) / (F3[[2]].x + ab3 - aw3) ≥ σ,
  ((B3[[3]] + F3[[3]]).x - aw3) / (F3[[3]].x + ab3 - aw3) ≥ σ,
  x1 + x2 + x3 == 1, x1 ≥ 0, x2 ≥ 0, x3 ≥ 0}, {x1, x2, x3, σ}]
```

```
{0.246064, {x1 → 0.443378, x2 → 0.317832, x3 → 0.23879, σ → 0.246064}}
```

Sakawa's direct method : Player II

Using properties of some constraints, the initial nonlinear programming problem for Player II, Sakawa (1983) retains the following equivalent programming problem

$$\max_{\mathbf{y},\lambda}\ \lambda$$

subject to

$$\frac{\sum_{j=1}^{n}(a_{k,ij} + a^{+}_{k,ij})y_j - \underline{a}_k}{\sum_{j=1}^{n} a^{+}_{k,ij}y_j + \bar{a}_k - \underline{a}_k} \leq \lambda,\ i \in \mathbb{N}_m,\ k \in \mathbb{N}_r$$

$$\mathbf{e}'\mathbf{y} = 1,\ (\mathbf{e},\ \mathbf{y} \in \mathbb{R}^n)$$

$$\mathbf{y} \geq 0.$$

The results for Player II are obtained with a degree of satisfaction of 58.5 per cent.

```
NMinimize[{λ,
  ((A1[[1]] + E1[[1]]).y - aw1) / (E1[[1]].y + ab1 - aw1) ≤ λ,
  ((A1[[2]] + E1[[2]]).y - aw1) / (E1[[2]].y + ab1 - aw1) ≤ λ,
  ((A1[[3]] + E1[[3]]).y - aw1) / (E1[[3]].y + ab1 - aw1) ≤ λ,
  ((A2[[1]] + E2[[1]]).y - aw2) / (E2[[1]].y + ab2 - aw2) ≤ λ,
  ((A2[[2]] + E2[[2]]).y - aw2) / (E2[[2]].y + ab2 - aw2) ≤ λ,
  ((A2[[3]] + E2[[3]]).y - aw2) / (E2[[3]].y + ab2 - aw2) ≤ λ,
  ((A3[[1]] + E3[[1]]).y - aw3) / (E3[[1]].y + ab3 - aw3) ≤ λ,
  ((A3[[2]] + E3[[2]]).y - aw3) / (E3[[2]].y + ab3 - aw3) ≤ λ,
  ((A3[[3]] + E3[[3]]).y - aw3) / (E3[[3]].y + ab3 - aw3) ≤ λ,
  y1 + y2 + y3 == 1, y1 ≥ 0, y2 ≥ 0, y3 ≥ 0}, {y1, y2, y3, λ}]
```

NMinimize::incst : NMinimize was unable to generate any initial points satisfying the inequality constraints
$\left\{-1. + 1.\,y2 + 1.\,y3 \leq 0,\ \frac{\ll 1 \gg}{\ll 1 \gg} - \lambda \leq 0,\ \ll 6 \gg,\ \ll 1 \gg - \lambda \leq 0,\ \frac{1 - 1.6\,y2 - 0.2\,(1. - \ll 1 \gg - 1.\,y3) + 6.1\,y3}{7.5 + 0.4\,y2 + 0.8\,(1. + \text{Times}[\ll 2 \gg] + \text{Times}[\ll 2 \gg]) + 0.1\,y3} - \lambda \leq 0\right\}$. The initial region specified may not contain any feasible points. Changing the initial region or specifying explicit initial points may provide a better solution. ≫

```
{0.585032, {y1 → 0.415622, y2 → 0.449909, y3 → 0.134469, λ → 0.585032}}
```

Q15. Fuzzy quadratic programming (1/3)

The symmetric approach by Zimmermann [23] may be used for solving fuzzy programming problems. For this approach, membership functions are defined, by using a given aspiration level of the decision maker (DM) for the objective, and accepted tolerances for the objective and the constraint functions. An equivalent crisp QP problem is obtained with a quadratic constraint. This particular QP problem can be solved by using van de Panne 's two-phase method [18]

D.1 Fuzzy QP problem

The fuzzy QP problem may be defined by a convex quadratic objective function together with a bounded feasible region such as [2]

$$\begin{aligned} \widetilde{\min}_{\mathbf{x}}\ \mathbf{c}'\mathbf{x} + \frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x} \\ \text{subject to} \\ \mathbf{A}_i\mathbf{x} \lesssim b_i,\ i \in \mathbb{N}_m \\ \mathbf{x} \geq 0, \end{aligned}$$

where $\mathbf{c},\ \mathbf{x} \in \mathbb{R}^n,\ \mathbf{b} \in \mathbb{R}^m,\ \mathbf{A} \in \mathbb{R}^{m\times n},\ \mathbf{Q} \in \mathbb{R}^{n\times n}$. The vector $\mathbf{A}_i$ denotes the ith row of matrix $\mathbf{A}$. The symmetric matrix $\mathbf{Q}$ is supposed to be positive semi definite.

D.2 Symmetric fuzzy QP problem

According to Zimmermann [23, 24], the symmetric version of the fuzzy QP problem is

$$\begin{aligned} \text{Find } \mathbf{x} \\ \text{such that} \\ \mathbf{c}'\mathbf{x} + \frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x} \gtrsim z_0, \\ \mathbf{A}_i\mathbf{x} \lesssim b_i,\ i \in \mathbb{N}_m \\ \mathbf{x} \geq 0, \end{aligned}$$

where $z_0 \in \mathbb{R}$ is the aspiration level of the DM and $p_0,\ p_i,\ i \in \mathbb{N}_m$ the tolerances for the objective and the set constraints, respectively. The membership function for the objective is defined by

$$\mu_0(z) = \begin{cases} 1, & z < z_0, \\ \frac{(z_0+p_0)-z}{p_0}, & z_0 \leq z \leq z_0 + p_0 \\ 0, & z \geq z_0 + p_0, \end{cases}$$

Q15. Fuzzy quadratic programming (2/3)

The membership function for the ith ($i \in \mathbb{N}_m$) constraint is also defined by

$$\mu_i(\mathbf{A}_i\mathbf{x}) = \begin{cases} 1, \ \mathbf{A}_i\mathbf{x} < b_i, \\ \frac{(b_i+p_i)-\mathbf{A}_i\mathbf{x}}{p_i}, \ b_i \leq \mathbf{A}_i\mathbf{x} \leq b_i + p_i \\ 0, \ \mathbf{A}_i\mathbf{x} \geq b_i + p_i. \end{cases}$$

An optimal solution is obtained by solving the crisp equivalent QP problem

$$\begin{aligned} &\text{Find } \alpha \\ &\text{such that} \\ \mathbf{c}'\mathbf{x} + \frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x} + \alpha p_0 &\leq z_0 + p_0, \\ \mathbf{A}_i\mathbf{x} + \alpha p_i &\leq b_i + p_i, \ i \in \mathbb{N}_m \\ \mathbf{x} &\geq 0, \quad \alpha \in [0,1]. \end{aligned}$$

Q15. Fuzzy quadratic programming (3/3)

The following numerical example is taken from Bector and Chandra [2], pp.77-78. The fuzzy symmetric QP problem is

$$
\begin{aligned}
&\text{Find } (x_1, x_2)\\
&\text{such that}\\
2x_1 + x_2 + 4x_1^2 + 4x_1x_2 + 2x_2^2 &\lesssim 51.88,\\
4x_1 + 5x_2 &\gtrsim 20,\\
5x_1 + 4x_2 &\gtrsim 20,\\
x_1 + x_2 &\lesssim 30,\\
x_1,\ x_2 &\geq 0.
\end{aligned}
$$

Let the tolerances be $p_0 = 2.12$, $p_1 = 2$, $p_2 = 1$, $p_3 = 3$, the equivalent crisp QP problem is

$$
\begin{aligned}
&\max \alpha\\
&\text{subject to}\\
2x_1 + x_2 + 4x_1^2 + 4x_1x_2 + 2x_2^2 + 2.12\alpha &\leq 54,\\
4x_1 + 5x_2 - 2\alpha &\geq 18,\\
5x_1 + 4x_2 - \alpha &\geq 19,\\
x_1 + x_2 + 3\alpha &\leq 33,\\
x_1,\ x_2 &\geq 0,\\
\alpha &\in [0, 1].
\end{aligned}
$$

The optimum solution of the QP problem, given by the multiplier method is $x_1^* = .9918$, $x_2^* = 3.7253$, $\alpha^* = .8599$. This result tells that the solution is obtained with a satisfaction level of 86 per cent.

Q16. Numerical example for the single objective FP problem

In the following two players example [6], Players I and II have two pure strategies. The goals of the two players are fuzzy. The payoffs are triangular fuzzy numbers. The LR-representations of the payoffs are the tensors $\widetilde{\mathbf{A}} \in \mathbb{R}^{2\times2\times3}$ and $\widetilde{\mathbf{B}} \in \mathbb{R}^{2\times2\times3}$ for Players I and II, respectively are

$$\widetilde{\mathbf{A}}_{LR} = \begin{pmatrix} (180,5,10) & (156,6,2) \\ (90,10,10) & (180,5,10) \end{pmatrix}$$

and

$$\widetilde{\mathbf{B}}_{LR} = \begin{pmatrix} (200,10,15) & (132,4,6) \\ (120,5,10) & (156,6,6) \end{pmatrix}$$

The right spread matrices are

$$\Delta_{\mathbf{A}} = \begin{pmatrix} 10 & 2 \\ 10 & 10 \end{pmatrix} \text{ and } \Delta_{\mathbf{B}} = \begin{pmatrix} 15 & 6 \\ 10 & 6 \end{pmatrix}$$

The optimal solutions of Player I are $x_1^* = .2366$ and $x_2^* = .7634$ w.r.t. a degree of attainment of the goal [7] of 75.3 per cent. The optimal solutions of Player II are $y_1^* = .2963$ and $y_2^* = .7037$ w.r.t. a degree of attainment of the goal of 39.4 per cent.

Q17. Numerical example for the multi-objective FP problem (1/2)

In the following two players example [11],[12]. Players I and II have respectively two and three pure strategies and three different objectives. The goals of the two players are fuzzy. The payoffs are triangular fuzzy numbers. The LR-representation of the payoffs are the tensors $\widetilde{\mathbf{A}}^k \in \mathbb{R}^{2\times3\times3}$, $k \in \mathbb{N}_3$ and $\widetilde{\mathbf{B}} \in \mathbb{R}^{2\times3\times3}$, $l \in \mathbb{N}_3$ for Players I and II respectively, are

$$\widetilde{\mathbf{A}}^1_{LR} = \begin{pmatrix} (1,.5,1) & (4,1,1) & (3,.5,1.5) \\ (2,1,1) & (4,.5,.5) & (1,1,1) \end{pmatrix}$$

$$\widetilde{\mathbf{A}}^2_{LR} = \begin{pmatrix} (4,.5,1) & (3,1,1) & (2,1,.5) \\ (1,1,1) & (5,1,.5) & (1,.5,1) \end{pmatrix}$$

$$\widetilde{\mathbf{A}}^3_{LR} = \begin{pmatrix} (2,1,1.5) & (0,0,1.5) & (1,.5,1) \\ (4,1.5,1.5) & (1,.5,.5) & (3,1,.5) \end{pmatrix}$$

and

$$\widetilde{\mathbf{B}}^1_{LR} = \begin{pmatrix} (0,0,1) & (2,1.5,1) & (2,1,1) \\ (5,.5,1) & (5,1,1) & (1,.5,.5) \end{pmatrix}$$

$$\widetilde{\mathbf{B}}^2_{LR} = \begin{pmatrix} (4,.5,1) & (2,1,1.5) & (5,1,.5) \\ (0,0,1) & (5,.5,.5) & (4,1.5,1) \end{pmatrix}$$

$$\widetilde{\mathbf{B}}^3_{LR} = \begin{pmatrix} (2,1,1.5) & (1,.5,1) & (4,1,1.5) \\ (1,.5,.5) & (0,0,1.5) & (1,1,1) \end{pmatrix}$$

The right spread matrices for Player I are

$$\Delta^1_{\mathbf{A}} = \begin{pmatrix} 1 & 1 & 1.5 \\ 1 & .5 & 1 \end{pmatrix}, \Delta^2_{\mathbf{A}} = \begin{pmatrix} 1 & 1 & .5 \\ 1 & .5 & 1 \end{pmatrix},$$

$$\Delta^3_{\mathbf{A}} = \begin{pmatrix} 1.5 & 1.5 & 1 \\ 1.5 & .5 & .5 \end{pmatrix}$$

The right spread matrices for Player II are

$$\Delta^1_{\mathbf{B}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & .5 \end{pmatrix}, \Delta^2_{\mathbf{B}} = \begin{pmatrix} 1 & 1.5 & .5 \\ 1 & .5 & 1 \end{pmatrix},$$

$$\Delta^3_{\mathbf{B}} = \begin{pmatrix} 1.5 & 1 & 1.5 \\ .5 & 1.5 & 1 \end{pmatrix}$$

Q17. Numerical example for the multi-objective FP problem (2/2)

The optimal solutions [13] of Player I are $x_1^* = .6438$ and $x_2^* = .3562$ w.r.t. a degree of attainment of the goal [14] of 58.5 per cent. The optimal solutions of Player II are $y_1^* = .5226$, $y_2^* = .3149$ and $y_3^* = .1625$ w.r.t. a degree of attainment of the goal of 52.5 per cent.

Q18. Application to economics

- Production model (Owen, 1975;Molina & Tejada, 2006;Nishizaki & Sakawa, 2000): multiple DMs pool resources to produce goods in a fuzzy environment; the total revenue from selling is maximized subject to constraints -> cooperative game theory
- Management of technology (MOT) (Chen & Larbani, 2006): optimal strategies in product development of nano-materials with MADM (Multiple attribute decision making)-> parametric bimatrix game
- Production inventory model (Park, 1987; Lee & Yao, 1998; Chang, 1999; Lin & Yao, 2000; Chen & Wang & Chang, 2006): imperfect production processes ; fuzzy inventory cost, fuzzy demand and production quantity, fuzzy quality of goods
- Oligopolistic competition (Greenhut & Greenhut & Mansur, 1995): fuzzy industry size, fuzzy interdependance,

Q19. Computational techniques for fuzzy programming problems

- **1965** (Zadeh): conception of the **Fuzzy Set Theory**
- **1976** (Cook): zero-sum games with multiple goals
- 1978- (Butnariu): fuzzy 2-person non cooperative games; solution concepts for n–persons fuzzy games
- 1980 (Dubois & Prade): LR-representation for computations
- 1983 (Chanas): parametric programming in FLP
- 1984 (Buckley): uncertainty of strategies and multiple fuzzy goals
- 1989 (Campos): solving matrix games with fuzzy payoffs based on ranking functions and LP methods.
- 1992- (Sakawa & Nishizaki): multiobjective matrix and bimatrix games with fuzzy payoffs and fuzzy goals
- 2001 (Sakawa): large scale interactive fuzzy multiobjective programming
- 2004 (Bector et al.): LP methods for solving matrix and bimatrix games

Q20. Main references

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