Generalized Differential-Difference Equations to Economic Dynamics and Control

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Abstract: Dynamic economic models generally consists in difference or differential behavioral equations. Several arguments are in favor of continuous time systems: the multiplicity of decisions overlapping in time, a more adequate formulation of market adjustments and distributed lag processes, the properties of estimators, etc. The type of dynamic equations also refer to historical and practical reasons. In some cases of the economic dynamics, mixed differential-difference equations (DDEs) may be more suitable to a wide range of economic models. The dynamics of the Kalecki's macroeconomic model is represented by a linear first-order DDE with constant coefficients, in the capital stock. Such a DDE, with constant or flexible lags, also occurs in the continuous time Solow's vintage capital growth model. This is due to the heterogeneity of goods and assets. In some qualitative study, the time delay is replaced by the Taylor series for a sufficiently small delay and a not too large higher-order derivative. DDEs with constant lags may be solved using Laplace transforms. Numerous techniques are also proposed for the solutions of DDEs, like the inverse scattering method, the Jacobian elliptic function method, numerical techniques, the differential transform method, etc. This study introduces the block diagram approach with application to reference economic models, with help of the powerful software *MATHEMATICA* 6.0. Specialized *MATHEMATICA* packages for signal processing are used for analyzing and solving, symbolically and numerically, the continuous and discrete systems, such as with "Control System Professional", "Polynomial Control Systems" and "SchematicSolver".

Key–Words: differential-difference equation, circuit analysis,...

1 Introduction

In some cases of the economic dynamics, mixed differential-difference equations (DDEs) may be more suitable to a wide range of economic models. The dynamics of the Kalecki's macroeconomic model is represented by a linear first-order DDE with constant coefficients, in the capital stock. Such a DDE, with constant or flexible lags, also occurs in the continuous time Solow's vintage capital growth model. This is due to the heterogeneity of goods and assets. In some qualitative study, the time delay is replaced by the Taylor series for a sufficiently small delay and a not too large higher-order derivative. DDEs with constant lags may be solved using Laplace transforms. Numerous techniques are also proposed for the solutions of DDEs, like the inverse scattering method, the Jacobian elliptic function method, numerical techniques, the differential transform method, etc. This study introduces the block diagram approach with application to reference economic models, with help of the powerful software *MATHEMATICA* 6.0. Specialized *MATHEMATICA* packages for signal processing are used for analyzing and solving, symbolically and numerically, the continuous and discrete systems, such as with "Control System Professional", "Polynomial Control Systems" and "SchematicSolver".

2 Elementary functional differential equation theory

2.1 Differential-difference equations

The linear form of a DDE of differential order m and difference order n with constant coefficients takes the form [1][7]

$$\sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}y^{(j)}(t-\omega_i) = f(t).$$

The subclass of the first-order DDE will then be written

$$a_0y'(t) + a_1y'(t-\omega) + b_0y(t) + b_1y(t-\omega) = f(t),$$

where $f(t)$ is assumed io be integrable and of boundary variation.

2.2 Solution methods

2.3 Basic equations

3 Continuous dynamic economic model with discrete delays

3.1 Business cycle Kalecki model

3.2 Business cycle Goodwin model

4 Continuous optimal control models with discrete delays

4.1 Ramsey's growth model

4.2 Solow's capital vintage model

A Laplace transform solution of equation $y'(t) + by(t-1) = 0$

The Laplace transform (denoted by $\mathcal{L}[.]$) method is used to solve a first-order linear DDE:

$$y'(t) + by(t-1) = 0,\ t > 0 \tag{1}$$

which boundary conditions are $y(t) = 0, t \in [-1, 0]$, and where b is a constant. We know that[2] $\mathcal{L}[y(t)] = \int_0^\infty y(t_1)e^{-st_1}dt_1 = Y(s),\ s \in \mathbb{C}$, and $\mathcal{L}[y'(t)] = sY(s) - y_0$. Multiplying (1) by e^{-st}, and integrating from 1 to infinity, we have also

$$\int_1^\infty y'(t)e^{-st}dt + b\int_1^\infty y(t-1)e^{-st}dt = 0. \tag{2}$$

Let us examine the two integrals of equation (2). Integrating by parts the first integral and assuming $y(t)e^{-st} \to 0$ as $t \to \infty$, we obtain

$$\int_1^\infty y'(t)e^{-st}dt = -y(1)e^{-s} + s\int_1^\infty y(t)e^{-st}dt. \tag{3}$$

Using a change of variable for the second integral yields

$$\begin{aligned} b\int_1^\infty y(t-1)e^{-st}dt &= b\int_{-1}^\infty y(t_1)e^{-s(t_1+1)}dt_1, \\ &= by_0e^{-s}\int_{-1}^0 e^{-st}dt + be^{-s}\int_{-1}^\infty y(t_1)e^{-st_1}dt_1, \\ &= by_0e^{-s}\left[-\frac{e^{st}}{s}\right]_{-1}^0 + be^{-s}Y(s), \\ &= \frac{by_0(1-e^{-s})}{s} + be^{-s}Y(s). \end{aligned} \tag{4}$$

Placing (3) and (4) into (2) yields

$$sY(s) - y_0 + \frac{by_0(1-e^{-s})}{s} + be^{-s}Y(s) = 0. \tag{5}$$

Solving (5) for $Y(s)$ and assuming that $s - e^{-s} \neq 0$, we get

$$Y(s) = \frac{y_0}{s} - \frac{by_0}{s(s+be^{-s})} \tag{6}$$

Theorem 1 *[33]. Let $f(t)$ be integrable over every finite interval such that (i) $\int_0^\infty f(t)e^{-st}$ converges absolutely on the real line $Re(s) = c$ and that (ii) $f(t)$ is of bounded variation in the neighborhood of t, then*

$$\oint_{(c)} F(s)e^{-st}ds = \frac{1}{2}\Big(f(t+0) - f(t-0)\Big),$$

where j denotes the pure imaginary $\sqrt{-1}$ and the LHS, a contour integral taken along along the line from $c - jT$ to $c + jT$ in the complex plane[3].

[1] The general form is

$$F\Big(t, y(t), y(t-\omega_1), \ldots, y(t-\omega_m), y'(t), y'(t-\omega_1), \ldots, y'(t-\omega_m), \ldots, y^{(n)}(t), y^{(n)}(t-\omega_1), \ldots, y^{(n)}(t-\omega_m)\Big) = 0$$

[2] The inverse Laplace transform is given for a suitable c by

$$\mathcal{L}^{-1}[Y(s)] = (2Pij)^{-1}\int_{c-j\infty}^{c+j\infty} Y(s)e^{st}ds = y(t)$$

.

[3] The contour integral is represented by

$$\oint_{(c)} F(s)e^{st}ds \equiv \lim_{T\to\infty}\frac{1}{2\pi j}\int_{c-jT}^{c+jT} F(s)e^{st}ds.$$

From (6), $Y(s)$ may also be expressed as

$$Y(s) = \frac{y_0}{s} - \frac{by_0}{s^2(1+b\frac{e^{-s}}{s})},$$
$$= y_0\left(\frac{1}{s} - \sum_{p=0}^{\infty}(-1)^p b^{p+1} e^{-ps} e^{-p-2}\right).$$

Applying the theorem 1, we have

$$y(t) = \oint_{(c)} y_0\left(\frac{1}{s} - \sum_{p=0}^{\infty}(-1)^p b^{p+1} e^{-s(t-p)}/e^{p+2}\right) ds$$
$$= y_0\left(\oint_{(c)} \frac{e^{st}}{s} ds - \sum_{p=0}^{\infty}(-1)^p b^{p+1}\oint_{(c)} e^{s(t-p)}/s^{p+2} ds\right). \quad (7)$$

Giving that (see [33], p.8)

$$\oint_{(c)} \frac{e^{hk}}{k^{n+1}} = \begin{cases} \frac{h^n}{n!} & , Re(h) > 0 \\ 0 & , Re(h) < 0, \end{cases}$$

we obtain the solution

$$y(t) = y_0\left(1 - \sum_{p=0}^{[t]}(-1)^p b^{p+1}\frac{(t-p)^{p+1}}{(p+1)!}\right),$$

where $[t]$ denotes the largest integer less or equal to t.

B Contour integral and definite integral solutions equivalence

The application of the Laplace transform method to the differential equation [4]

$$y'(t) + by(t) = f(t),\ t > 0, y(0) = y_0$$

leads to a contour integral representation solution of the form

$$y(t) = \oint_{(c)}\left(\frac{y_0 + \int_0^t f(t_1)e^{-b(t-t_1)}dt_1}{s+b}\right)e^{st} ds,\ t \geq 0. \quad (8)$$

However, the method of the integrating factor leads to definite real integral

$$y(t) = y_0 e^{-bt} + \int_0^t e^{-b(t-t_1)}dt_1,\ t \geq 0. \quad (9)$$

[4]This example is taken from [7], p. 71, with $a_0 = 1$ and $b_0 \equiv b$.

Let us show the equivalence of the two results.
Proof. According to (8), we have

$$y(t) = y_0\oint_{(c)} \frac{e^{st}}{s+b} ds + \int_{(c)} \frac{e^{st}\oint_0^{\infty} f(t_1)e^{-st_1}dt_1}{s+b} ds,$$
$$= y_0\mathcal{L}[e^{-bt}] + \mathcal{L}[f(t_1)e^{-b(t-t_1)}dt_1].$$

Then the solution expression (9) in the form of a definite integral follows from the solution expression (8) of a contour integral. ■

C Tinbergen's shipbuilding cycle

The Tinbergen's equation [39] is of the form [5]

$$y'(t) = -by(t-1),\ b > 0, t > \theta. \quad (10)$$

We also assume $y(t) = h(t),\ t \in [0, \theta)$, where $h(t)$ is some given function. In this application to the shipbuilding industry, y denotes the deviation of the tonnage from a mean value and θ a given constant construction time. In this equation, Tinbergen assumes the rate of new shipbuilding to be proportional to a delayed tonnage deviation, with a one to two years delay θ and a ranged intensity reaction $b \in [\frac{1}{2}, 1]$. An endogenous cycle is found for the shipbuilding industry, with a period of about 8 years: 7 years 6 months for $\theta = 1$ and 8 years 9 months for $\theta = 2$.

C.1 Characteristic equation

Let the form of the unknown function be $y(t) = e^{\rho t}$, the characteristic equation from (10) is

$$D(\rho) \equiv \rho + be^{-\rho\theta} = 0,\ \rho \in \mathbb{C}, \quad (11)$$

where $\rho = \beta + \alpha j$, $j = \sqrt{-1}$. Inserting (11) into (10) and separating the real and imaginary parts, we obtain the system

$$\begin{aligned} \cos\ u &= \frac{-v}{\theta b}e^v, \\ \frac{\sin\ u}{u} &= \frac{1}{\theta b}e^v, \end{aligned}$$

where $u \equiv \alpha\theta$ and $v \equiv \beta\theta$. Eliminating v, we obtain an even function $f(u)$ in which the structural coefficients θ, b are not explicit. We have

$$f(u) = \frac{u}{\tan\ u} + \ln\frac{\sin\ u}{u} = C, \quad (12)$$

[5]A nonlinear DDE version is given by [33]

$$y'(t) = -by(t-1) - \varepsilon y^3(t-1),\ \varepsilon, b > 0$$

.

where $C \equiv -\ln(\theta b)$. A further analysis of the characteristic equation is given by Pinney [33] by means of the (x,k)-root plateau in the parameter space[6]. The properties of the caracteristic equation are summarized in Figure 1.

existence of roots	kind of roots	range of b
no root	pseudo−positive	$(0,\frac{\pi}{2})$
one root	pseudo−positive	$(-\frac{3\pi}{2},0)$
2n roots	pseudo−positive	$((2n-\frac{3}{2})\pi,(2n+\frac{1}{2})\pi)$, n∈N
2n+1 roots	pseudo−positive	$(-(2n+\frac{3}{2})\pi,-(2n-\frac{1}{2})\pi)$, n∈N

Figure 1: Exponential behavior

C.2 Existence of exponential components

Let $z \equiv \rho\theta$, (11) may be expressed as

$$-\frac{z}{\theta b} = e^{-z}. \tag{13}$$

The two parts of (11) are plotted [7] in Figure 2. The condition for tangency of the two curves $1/(\theta b) = e^{-z}$ is $z = \ln(\theta b)$. Inserting in (13), we get $C = 1$. The solution of the DDE (10) is a pure exponential [8] of the type

$$y(t) = (C_1 + C_2 t)e^{\frac{z}{\theta}t}.$$

For $C > 1$, the solutions are composed of two exponentials in the period ranges

$$T \in \left(\frac{\theta}{k}, \frac{\theta}{k-\frac{1}{2}}\right),\ k \in \mathbb{N}.$$

C.3 Existence of cyclical components

A cycle corresponds to each real solution of (12) when $C < 1$. The two sides of this equation are represented in Figure 3. Real branches of $f(u)$ decrease monotically in all the intervals $[k2\pi, (2k+1)\pi],\ k \in \mathbb{N}_0$.

[6]According to this concept, the parameters may be chosen in order to achieve some desired properties for the system. Let $\rho = x + jy$,the (x,k)-root plateau represents the sets of parameter values for which the caracteristic equation has k pseudo roots greater than x. The equations of the (x,k)-root plateau on the b-line are

$$\begin{aligned} Re(D(\rho)) &= x + be^{-x}\cos y, \\ Im(D(\rho)) &= y - be^{-x}\sin y. \end{aligned}$$

[7]The figure shows a state of a dynamic interactive Mathematica plotting with automatic sliders and controls.

[8]A degenerate cycle with infinite period.

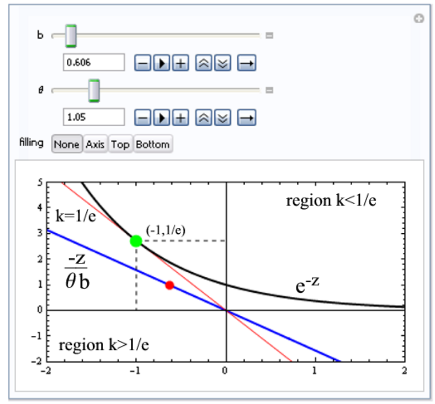

Figure 2: Exponential behavior

According to $u = \alpha\theta$ and $\alpha = \frac{2\pi}{T}$, the corresponding period ranges are

$$T \in \left(\frac{\theta}{k}, \frac{\theta}{k-\frac{1}{2}}\right),\ k \in \mathbb{N}_0.$$

The sine curves may be damped or undamped. A dictinction is made between the major cycle of the first period and the minor cycles

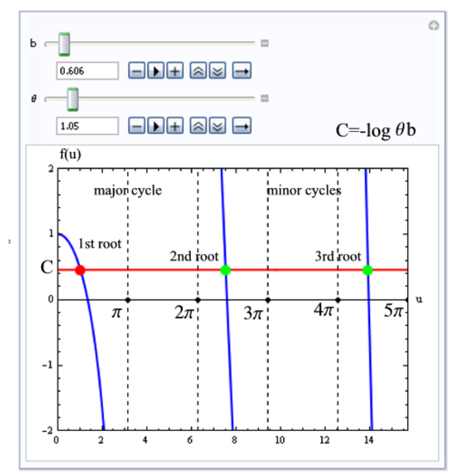

Figure 3: Cyclical behavior

C.4 Stability of the components

The stability regions are shown in tne (b, θ)- parameter plane (Figure 4) The corresponding patterns of components are shown in Figure 5.

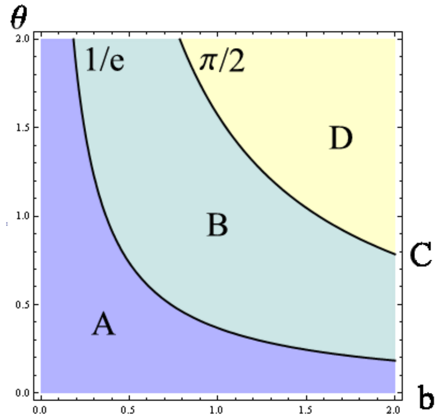

Figure 4: Stability diagram

region	k=θ b	period T	component pattern
A	k<$\frac{1}{e}$	∞	monotonic
B	k$\in$($\frac{1}{e}$,$\frac{\pi}{2}$)	T>4θ	damped oscillatory
C	k=$\frac{\pi}{2}$	T=4θ	pure oscillatory
D	k>$\frac{\pi}{2}$	T$\in$(2θ,4θ)	nondamped oscillatory

Figure 5: Pattern of the components

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