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Fuzzy Conflict Games in Economics and Management: single objective fuzzy bi-matrix games

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Abstract In the real game situations, the possible values of parameters are imprecisely known to the experts, all data of the game are not exactly known to the players, and information is often lacking. Imprecision on the environment, preferences, payoffs and moves of other players, may be of different types, but not only the probabilistic type of the Bayesian games. In the probabilistic approaches of uncertainty, events or statements are assumed to be well defined. On the contrary, the Zadeh's fuzziness concept extends the imprecision or vagueness appreciations to that events and statements. The theory of fuzzy sets has been extensively applied to a variety of domains in soft computing, modeling and decision making. This contribution introduces these attractive techniques with numerical applications to economic single-objective bi-matrix games. The computations are carried out using the software $MATHEMATICA^{®}$7.0.1.

Keywords: fuzzy logic game; fuzzy linear/quadratic programming problem; extension principle; function principle.

1. Introduction

1.1. Fuzzy approach

In the real game situations, the possible values of parameters are imprecisely known to the experts and all data of the game are not exactly known to the players. Imprecision on the environment, preferences, payoffs and moves of other players, may be of different types, but not only the probabilistic type of the Bayesian games. The fuzzy sense of imprecision, introduces a degree of membership for each element of a given set. In the classical crisp collection of elements X, each element x can belong to or not to the set. For a fuzzy set $\tilde{A}$, the characteristic function allows various degrees of membership $\mu_{\tilde{A}}(x)$ for $x \in X$, where X denotes the universe space. Therefore, the fuzzy set $\tilde{A}$ in X is a set of ordered pairs $\{\big(x, \mu_{\tilde{A}}(x)\big)|x \in X\}$. This introduction presents elements on the fuzzy games theory, applications of fuzzy games to economic and management problems, and recalls the equivalence we have between games and technical programming problems.

1.2. Fuzzy games history

Research on fuzzy games have been developed rapidly since the mid 1970s (Zadeh et al., 1974- Negoită and Ralescu, 1975- Butnariu, 1978- Aubin, 1979). The study by (Nishizaki and Sakawa, 2000a) formulates a linear programming (LP) problem with fuzzy (triangular) parameters and a fuzzy goal of each coalition of players. In the fuzzy programming problem, the decision maker (DM) may know the cost coefficients in the objective function, whereas the payoffs in the constraints would stay imprecise (Campos, 1989). Fuzzy logic games (FLG) are a component of a larger class of combinatorial games and also belong to the so-called ”soft computing” which combines fuzzy logic, neural networks and evolutionary programming (Aubin, 1979, 1981). In FLGs, the decision making is re-formulated in an uncertain (fuzzy) environment : the decision makers are confronted with fuzzy constraints, fuzzy utility maximization and also fuzziness about the moves of the competitors (Campos, 1989- Vodyottema et al., 2004, de Wilde, 2004). Moreover, cooperative FLGs are describing coalitions, where the n players associate a certain rate to their participation (Mareš, 2001). Such games are defined on fuzzy subsets of the whole set of n players (Aubin, 1979, 1981- Garagic and Cruz, 2003- Azrieli and Lehrer, 2007- Hwang, 2007). The axiomatic analysis by (Billot, 1992) reformulates the basic microeconomic theory to deal with fuzzy choice and preferences. The fuzzy preference operator $\succsim$ is defined by $\succsim: X^2 \mapsto [0,1]$, X denoting the set of alternatives. Hence $x \succsim y$ will be interpreted as the degree to which x is at least as good as y. The resolution method consists in introducing tolerance levels which allow the violation of each constraint (Delgado et al., 1989). The concepts of equilibrium may be based on Zimmermann's approach, in two steps for solving LP multi-objective problems with fuzzy goals (Bellman and Zadeh, 1970- Chanas, 1978- Lai and Hwang, 1992- Kim and Lee, 2001- Bector and Chandra, 2005- Kacher and Larbani, 2008). In the reality, the decision makers with conflicting interests, are faced to multiple attributes such as costs, time and productivity. For such problems, methods and applications have been developed in (Zimmermann, 1978- Nishizaki and Sakawa, 2000a, 2001 - Sakawa, 2000). These models are based on the maximin and the minimax principles of the matrix game theory. The equilibrium solutions correspond to players trying to maximize a degree of attainment of the fuzzy goals. The aggregation of all the fuzzy sets in the multi-objective models use the fuzzy decision rule by Bellman and Zadeh (Lai and Hwang, 1994- Chen, 2002- Keller, 2009a,b).

1.3. Fuzzy games to economics and management

Fuzzy games have been applied to a wide range of subjects in economics and management modeling, such as: linear production models, inventory models, oligopolistic competition markets, and management of technology (MOT).

Linear production model. The pioneered production model by (Owen, 1975) is describing a cooperative game in which the players pool resources to produce finished goods. The goods are next sold at the market price. In the resulting LP production model, the total revenue is maximized subject to the resource constraints. The Owen's production model has been extended to fuzzy situations, notably by (Nishizaki and Sakawa, 2000b,c- Chen et al., 2007). The parameters involved in the objective and in the constraints of the LP problem are fuzzy numbers which reflect the imprecise knowledge of experts. The DMs maximize the total revenue from selling, subject to the resource constraints, and in absence of demand limitation. A parametric programming approach is used to solving the linear production model. Nishisaki and Sakawa (Nishizaki and Sakawa, 2000b) also prove the existence and non-emptiness of the α-core of the fuzzy game. In (Nishizaki and Sakawa, 2000c), two solution concepts based on fuzzy goals are defined: one is defined by maximizing the minimal fuzzy goal and the other by maximizing the sum of fuzzy goals. Molina and Tejada (Molina and Tejada, 2006) analyze a linear production with committee control to allow players to graduate their cooperation willingness. The resources are controlled by committees of players in a fuzzy context.

Production inventory model. The studies by (Park, 1987- Lee and Yao, 1996- Yao and Lee, 1998- Lin and Yao, 2000) aim the optimality of the stock quantity of the inventory with back orders. The economic orders quantities are fuzzy numbers (FNs). As a result, the total cost is found to be higher than in the crisp model. In (Chen et al., 1996), the fuzzy context is extended to demand, order costs and back order costs. The computation of $\tilde{B} = f(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ with trapezoidal FNs $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ uses the Chen's function principle (instead of the Zadeh's extension principle). In their study, Chen et al. (Chen et al., 1996) propose a production inventory model with imperfect products to the electronic industry. The fuzzy arithmetic operations with trapezoidal FNs also use the function principle.

Oligopolistic market model. In their study, (Greenhut et al., 1995) propose a fuzzy approach to the oligopolistic market. According to this approach, the membership function expresses the degree to which a firm belongs to the oligopolistic market.

Management of technology. The Management of Technology (MOT) is concerned with the identification and selection of technologies, innovation management, transfer and licensing, R & D. In the MOT, the problem of determining optimal strategies is transformed to a problem of fitting Nash equilibria of a bi-matrix game. The study by (Chen and Larbani, 2006) approaches the product development of nano materials in a matrix game model for fuzzy multiple attributes decision making problems.

1.4. Equivalence theorems

Two Players I and II have mixed strategies given by the m-dimensional vector $\mathbf{x}$ and the n-dimensional vector $\mathbf{y}$, respectively. The payoffs of Players I and II are the $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, respectively. Let $\mathbf{e}_m$ be an m-dimensional vector of ones, $\mathbf{e}_n$ having a dimension n. The objective of Player I will be: $\{\max_{\mathbf{x}}\ \mathbf{x'Ay}$ subject to $\mathbf{e'}_m\mathbf{x} = 1,\ \mathbf{x} \geq 0\}$. The objective of Player II will then be: $\{\max_{\mathbf{y}}\ \mathbf{x'By}$ subject to $\mathbf{e}'_n\mathbf{y} = 1,\ \mathbf{y} \geq 0\}$. Following (Mangasarian and Stone, 1964), a bi-matrix game is shown to be equivalent to a quadratic programming (QP) problem and a zero-sum game to a LP problem.

Equivalence to QP problems

Definition 1. (Nash equilibrium). A Nash equilibrium point is a pair of strategies $(\mathbf{x}^*, \mathbf{y}^*)$ such that the objectives of the two players are full filled simultaneously. We have

$$\mathbf{x}^{*'}\mathbf{A}\mathbf{y}^* = \max_{\mathbf{x}}\{\mathbf{x'A}\mathbf{y}^* | \mathbf{e}_m.\mathbf{x} = 1, \mathbf{x} \geq 0\}$$
$$\mathbf{x}^{*'}\mathbf{B}\mathbf{y}^* = \max_{\mathbf{y}}\{\mathbf{x}^{*'}\mathbf{B}\mathbf{y} | \mathbf{e}_n.\mathbf{y} = 1, \mathbf{y} \geq 0\}$$

Applying the Kuhn-Tucker (K-T) necessary and sufficient conditions, we set the Equivalence Theorem (Mangasarian and Stone, 1964- Lemke and Howson, 1964- van der Panne, 1966- Shimizu and Aiyoshi, 1980):

Theorem 1. *(Equivalence Theorem). A necessary and sufficient condition that* $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ *be an equilibrium point is to correspond to the solution of the QP prob-*

lem

$$\max_{\boldsymbol{x},\boldsymbol{y},\alpha,\beta} \ \boldsymbol{x}'(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{y} - \alpha - \beta$$

$$\begin{aligned} &\textit{subject to} \\ \boldsymbol{B}'\boldsymbol{x} &\leq \beta \boldsymbol{e}_n, \\ \boldsymbol{A}\boldsymbol{y} &\leq \alpha \boldsymbol{e}_m, \\ \boldsymbol{e}'_m \boldsymbol{x} &= 1, \\ \boldsymbol{e}'_n \boldsymbol{y} &= 1, \\ \boldsymbol{x} \geq 0,\ \boldsymbol{y} &\geq 0, \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ are the negative of the multipliers associated with the constraints.

`Proof`: see (Mangasarian and Stone, 1964), pages 350-351.□

Equivalence to LP problems In zero-sum games we have $\mathbf{B} = -\mathbf{A}$ with $\gamma = -\beta$. The QP problem degenerates to two dual problems (see Fig.1).

$$\begin{aligned} \max_{\mathbf{x},\gamma} \quad & \gamma \\ \text{subject to} \\ -\mathbf{A}'\mathbf{x} &\leq -\gamma \mathbf{e}_n, \\ \mathbf{e}'_m \mathbf{x} &= 1, \\ \mathbf{x} &\geq 0. \end{aligned} \qquad \textbf{and} \qquad \begin{aligned} \min_{\mathbf{y},\alpha} \quad & \alpha \\ \text{subject to} \\ \mathbf{A}\mathbf{y} &\leq \alpha \mathbf{e}_m, \\ \mathbf{e}'_n \mathbf{y} &= 1, \\ \mathbf{y} &\geq 0. \end{aligned}$$

Figure1. LP dual problems

A numerical example is shown in Fig.1. In this example, the payoff matrices are

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

and the strategies are $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. The application of the Equivalence Theorem [1] gives the optimal strategies $\mathbf{x}^* = (.6, .4)$ and $\mathbf{y}^* = (.6, .2)$.

2. Fuzzy data environment and fuzzy games

Besides of the usual "True" and "False" binary statements, there are also vague (or fuzzy) statements in the real world of the decision making. The linguistic statements may be : "possible", "almost sure", "hardly fulfilled","approximately equal to", "considerable larger to", etc.

[1] Algorithms for solving two-person games are presented in (Canty, 2003-Engwerda, 2005).

```
Clear[A, B];
A := {{2, -1}, {-1, 1}}; B := {{1, -1}, {-1, 2}}; x := {x1, x2}; y := {y1, y2}; e := {1, 1}
Timing[Maximize[Rationalize[
   {x.(A + B).y - α - β,
    A[[1]].y ≤ α && A[[2]].y ≤ α &&
     Transpose[B][[1]].x ≤ β && Transpose[B][[2]].x ≤ β &&
     e.x == 1 && e.y == 1 &&
     x1 ≥ 0 && x2 ≥ 0 && y1 ≥ 0 && y2 ≥ 0}],
  {x1, x2, y1, y2, α, β}]]

{61.735, {0, {x1 → 3/5, x2 → 2/5, y1 → 2/5, y2 → 3/5, α → 1/5, β → 1/5}}}

N[%]
{61.735, {0., {x1 → 0.6, x2 → 0.4, y1 → 0.4, y2 → 0.6, α → 0.2, β → 0.2}}}
```

Figure2. Solving games with *MATHEMATICA*

2.1. Fuzzy number representations

Definition 2. (Membership function of a fuzzy number). Any vague statement $\tilde{S}$ is considered as a fuzzy subset of a universe space X with the membership function $\mu_{\tilde{S}} : X \mapsto [0,1]$. For any $x \in X$: $\mu_{\tilde{S}}(x) = 1$ means $\tilde{S}$ is "True" for x; $\mu_{\tilde{S}}(x) = 0$ means $\tilde{S}$ is "False" for x, and $0 < \mu_{\tilde{S}}(x) < 1$ means $\tilde{S}$ is "possible" for x with the degree of possibility $\mu_{\tilde{S}}(x)$. This function is called the membership function (MF) of the fuzzy number (FN).

Let the MF of the fuzzy $\tilde{A}$ be piecewise continuous triangular shaped. The fuzzy $\tilde{A}$ is said convex normalized. The support of $\tilde{A}$ is such that *supp* $\tilde{A} = \{x \in X | \mu_{\tilde{A}}(x) = 0\}$. The height of $\tilde{A}$ is such that *hgt* $\tilde{A} = \sup_x \mu_{\tilde{A}}(x)$. The crossover points are defined by $c = \{x | \mu_{\tilde{A}}(x) = \frac{1}{2}\}$. The α-cuts of $\tilde{A}$ gives the crisp set of elements with at least the degree α, such that ${}^{\alpha}A = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}$ (see Fig.3).

LR-representation

Definition 3. (LR-type fuzzy number). A fuzzy $\tilde{A}_{LR} = \left(\bar{a}, \delta_a^-,\ \delta_a^+\right)$ is LR-type if there exist reference functions L (for left) and R (for right), and positive scalars δ_a^-, δ_a^+ such that

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{\bar{a}-x}{\delta_a^-}), & x \leq \bar{a} \\ R(\frac{x-\bar{a}}{\delta_a^+}), & x \geq \bar{a}, \end{cases}$$

where $\bar{a}$ denotes the "mean value" and δ_a^-, δ_a^+ the left and right spreads. For the example in Fig.4, we have the reference functions

$$L(x) = \frac{1}{1+x^2} \text{ and } R(x) = \frac{1}{1+2|x|}.$$

The LR-representation (Dubois and Prade, 1980) increases notably the computational efficiency.

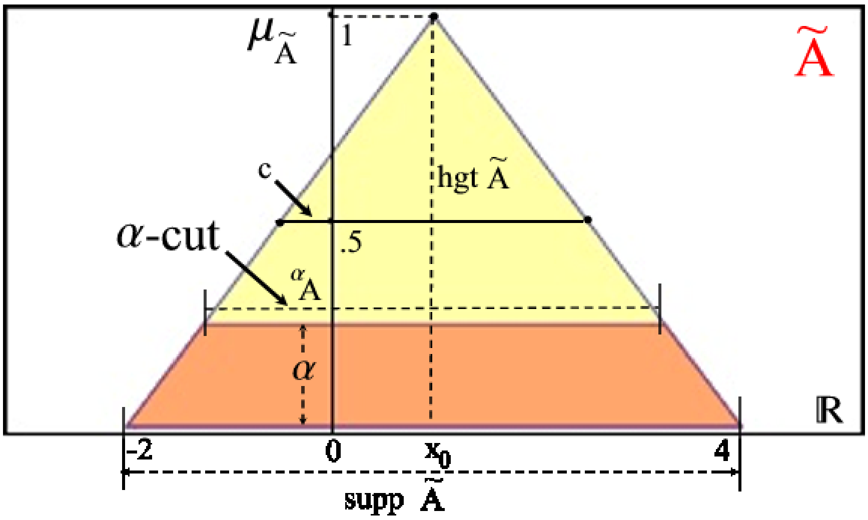

Figure3. Fuzzy number elements

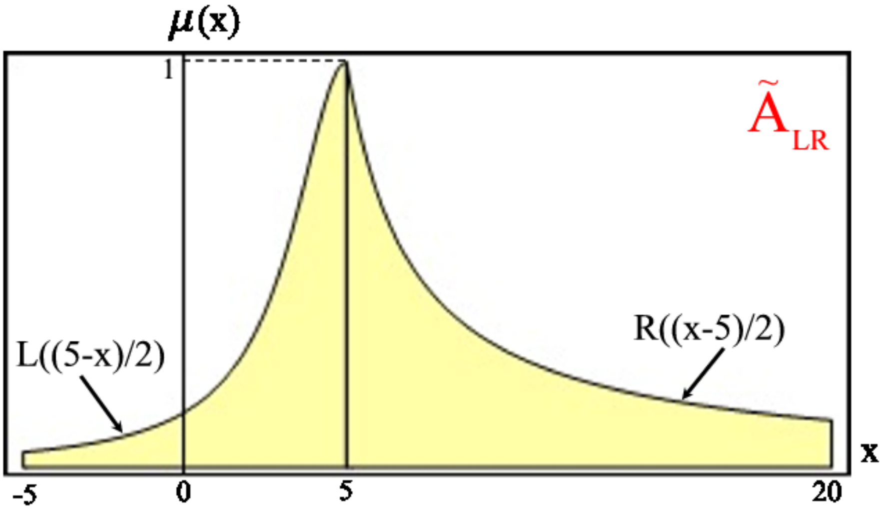

Figure4. LR-representation of a FN

Membership shapes. The MFs may take one of the basic forms in Fig.5. The Figure A is an increasing ramp MF type. The Figure B is a sigmoidal MF for which a parameter controls the slope at the crossover point $(c, \mu(c))$. The Figure C represents a bell-shaped fuzzy set, centered at c with crossover points $c-w$ and $c+w$, and with slope $s/2w$ at the crossover points. The trapezoidal and the triangular forms (Figures D and E) are often used, because of their simplicity.

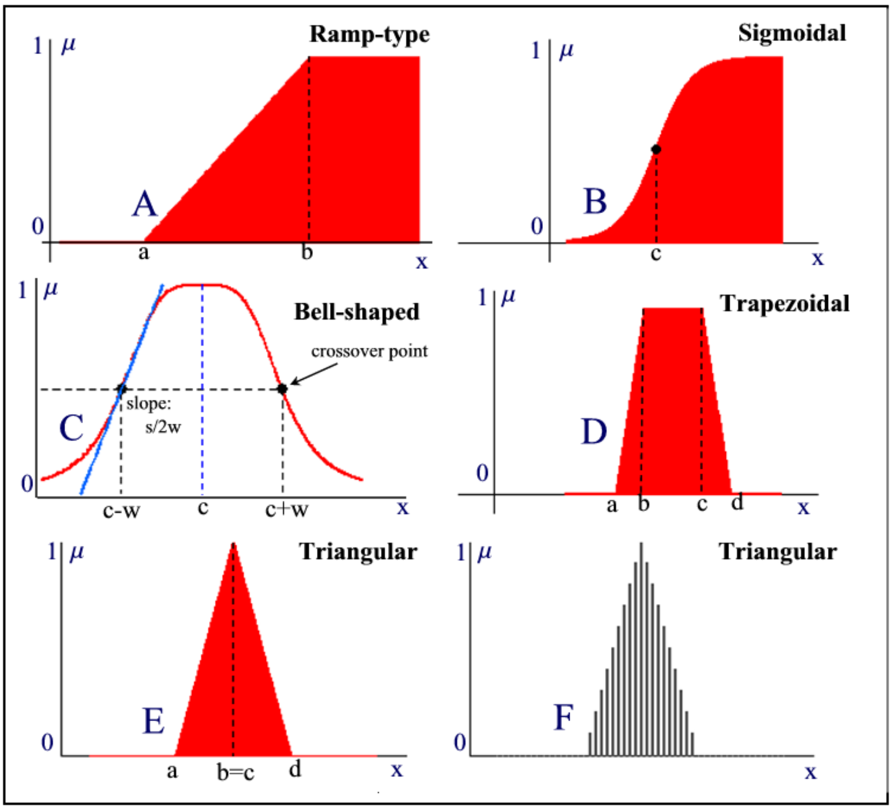

Figure5. Basic shapes for membership functions

2.2. Arithmetic operations on fuzzy numbers

The extension principle is a method of calculating the MF of the output from the MFs of the input fuzzy quantities. More precisely, let the symbol $* : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ with $* \in \{+,-,.,/\}$, a binary operation over real numbers. Then, it can be extended to the operation $\circledast$ over the set $\mathfrak{F}(\mathbb{R})$ of fuzzy quantities (see Appendix B).

Extension Principle [2]

[2] On the contrary to the convolution form of the extension principle, Chen's rule is a useful pointwise multiplication for trapezoidal MFs. Let F be a mapping from n-dimension FNs belonging to the trapezoidal family $\tilde{A}_i = (a_i, b_i, c_i, d_i; w_i)$ where $w_i = \max_x \mu_{\tilde{A}_i}(x)$ $i \in \mathbb{N}_n$. The fuzzy $\tilde{B}$ in $\mathbb{R}$ is $\tilde{B} = F(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) = (r, s, t, u : w)$. The determination

Theorem 2. *(Extension principle). Denote for $\tilde{a},\ \tilde{b} \in \mathfrak{F}(\mathbb{R})$ the quantity $\tilde{c} = \tilde{a} \circledast \tilde{b}$, then the MF μ_c is derived from the continuous MFs μ_a and μ_b by the expression*

$$\mu_{\tilde{a}\circledast\tilde{b}}(z) = \sup_{z=x*y} \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\}.$$

This formula tells that the possibility that the fuzzy quantity $\tilde{c} = \tilde{a} \circledast \tilde{b}$ achieves $z \in \mathbb{R}$ is as great as the most possible of the real $x,\ y$ such that $z = x * y$, where the $a,\ b$ take the values x, y respectively. For the addition, we have the ordinary convolution

$$\mu_{\tilde{a} \bigoplus \tilde{b}}(z) = \int_0^z \mu_{\tilde{a}}(x)\mu_{\tilde{b}}(z-x)dx.$$

Example. Let the MFs of the fuzzy $\tilde{a}$ and $\tilde{b}$ be defined in Fig.6. The α-cuts are ${}^{\alpha}\tilde{a} =$

$$\mu_{\tilde{a}}(x) = \begin{cases} x-1, & x \in [1,2], \\ 3-x, & x \in [2,3], \\ 0, & \texttt{otherwise}. \end{cases} \quad \textbf{and} \quad \mu_{\tilde{b}}(x) = \begin{cases} x-4, & x \in [5,7], \\ (7-x)/2, & x \in [7,10], \\ 0, & \texttt{otherwise}. \end{cases}$$

Figure6. Fuzzy numbers $\tilde{a}$ and $\tilde{b}$

$[\alpha+1, 3-\alpha]$ and ${}^{\alpha}\tilde{b} = [\alpha+4, 7-2\alpha]$. Then, we have ${}^{\alpha}\tilde{c} = {}^{\alpha}(\tilde{a} \bigoplus \tilde{b}) = {}^{\alpha}\tilde{a} + {}^{\alpha}\tilde{b} = [2\alpha+5, 10-3\alpha]$. Solving in α, we obtain

$$\mu_{\tilde{a} \bigoplus \tilde{b}}(x) = \begin{cases} (x-5)/2, & x \in [5,7], \\ (10-x)/3, & x \in [7,10], \\ 0, & \texttt{otherwise}. \end{cases}$$

Theorem 3. *(Addition of LR-type fuzzy numbers). Let $\tilde{a}$ and $\tilde{b}$ two FNs of LR-type $\tilde{a} = (\bar{a}, \delta_a^-, \delta_a^+)_{LR}$ and $\tilde{b} = (\bar{b}, \delta_b^-, \delta_b^+)_{LR}$, then $\tilde{a} \bigoplus \tilde{b} = (\bar{a}+\bar{b}, \delta_a^- + \delta_b^-, \delta_a^+ + \delta_b^+)_{LR}$.*

For $\tilde{a} = (2,1,1)_{LR}$ and $\tilde{b} = (5,1,2)_{LR}$, we simply have $\tilde{a} \bigoplus \tilde{b} = (7,2,3)_{LR}$. The extended addition is illustrated in Fig.7.

2.3. Standard fuzzy games

Standard fuzzy games are LP problems with fuzzy constraints.To solve the LP problem, the fuzzy constraints must be converted into crisp inequalities by using some ranking functions. One auxiliary problem is then to be solved. We will consider three situations for the game. In the first case, the resources of the production problem are imprecise (fuzzy) to the DM. In the second case, the technical coefficients are nonsymmetric triangular FNs. In the third case, the fuzzy model is extended to soft constraints, when the DM allows some violation in the accomplishment of the constraints. A numerical example illustrates a general situation, where the resources, the technical coefficients and the inequalities are all imprecise.

of the trapezoidal parameters r, s, t, u and w is given in Appendix A, for $\tilde{A} * \tilde{B}$ with $* \in \{+, -, ., /\}$.

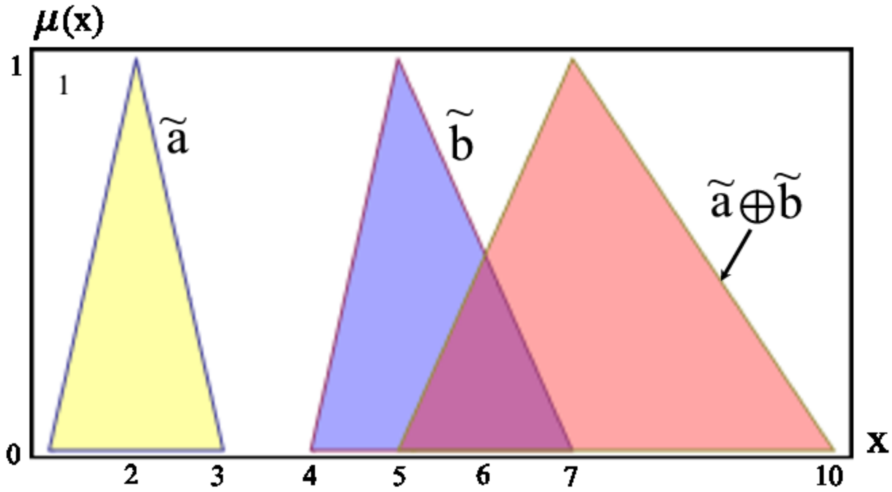

Figure7. Extended addition of fuzzy sets

Case 1: LP problem with fuzzy resources . Let the maximizing LP problem with fuzzy (imprecise) resources $\tilde{\mathbf{b}}$ be

$$\begin{aligned} \max_{\mathbf{x}} \ \mathbf{c}^{\top}.\mathbf{x},\ (\mathbf{c},\mathbf{x}\in\mathbb{R}^n) \\ \text{subject to} \\ \mathbf{A}_i.\mathbf{x}\leq\tilde{b}_i\ (i\in\mathbb{N}_m), \\ \mathbf{x}\geq 0. \end{aligned}$$

In a production scheduling problem, the vector $\mathbf{c}$ denotes n costs, the $m\times n$ matrix $\mathbf{A}$ technical coefficients, the vector $\mathbf{b}$ m resources and the vector $\mathbf{x}$ the n variables. The solution algorithm of the Zimmermann's symmetric method consists in the following steps:

Step 1 : Definition of the memberships and determination of the fuzzy feasible set . Let the ith resource b_i being defined by the interval $[b_i, b_i+p_i]$ with tolerance p_i.The memberships of the fuzzy $\tilde{b}_i$'s are of the piecewise linear type ($x\in\mathbb{R}$). We have

$$\mu_i(x)=\begin{cases} 1, x\ \geq b_i \\ 1-\frac{x-b_i}{p_i},\ b_i<x<b_i+p_i \\ 0,\ x\geq b_i+p_i \end{cases}$$

The degree $D_i(\mathbf{x})$ to which $\mathbf{x}$ satisfies the ith constraint is then $\mu_i(\mathbf{A}_i.\mathbf{x})$. All the μ_i's define fuzzy sets on $\mathbb{R}^n$ and the MF of fuzzy feasible set is

$$\bigwedge_{i=1}^{m}\mu_i(\mathbf{A}_i.\mathbf{x}).$$

Step 2 : Definition of the fuzzy optimal values . The objective admits a lower and an upper bound respectively equal to

$$\max\left\{z_l=\mathbf{c'}.\mathbf{x}\mid\mathbf{A}_i.\mathbf{x}\leq b_i\ (i\in\mathbb{N}_m),\ \mathbf{x}\geq 0\right\}$$

and

$$\max \left\{ z_u = \mathbf{c'.x} \mid \mathbf{A}_i.\mathbf{x} \leq b_i + p_i \ (i \in \mathbb{N}_m), \ \mathbf{x} \geq 0 \right\}.$$

The MF of the single objective G is defined ($x \in \mathbb{R}$) by

$$\mu_G(x) = \begin{cases} 1, \ \geq b_i \\ \frac{\mathbf{c'.x} - z_l}{z_u - z_l}, \ z_l < \mathbf{c'.x} < z_l \\ 0, \ \text{otherwise} \end{cases}$$

Step 3: Solution by using the max-min operator . The problem

$$\max \left((\bigcap_{i=1}^{m} D_i) \bigcap G \right)(\mathbf{x})$$

is described by the equivalent crisp LP problem

$$\begin{aligned} \max_{\mathbf{x},\lambda} \ \lambda \\ \text{subject to} \\ \mu_G(\mathbf{x}) = \frac{\mathbf{c'.x} - z_l}{z_u - z_l} \geq \lambda \ , \\ \mu_i(\mathbf{x}) = 1 - \frac{\mathbf{A}_i.\mathbf{x} - b_i}{p_i} \geq \lambda, \ (i \in \mathbb{N}_m), \\ \mathbf{x}, \lambda \geq 0. \end{aligned}$$

Case 2: LP problem with fuzzy technical coefficients . Suppose that all the coefficients of the constraints are nonsymmetric triangular FNs (TFNs)

$$\tilde{a}_{ij} = (a_{ij}, a_{ij} - \underline{a}_{ij}, a_{ij} + \bar{a}_{ij}), \ \texttt{and} \ \tilde{b}_i = (b_i, b_i - \underline{b}_{ij}, b_i + \bar{b}_i).$$

According to the operations on the TFNs (using a simple partial order such that $\tilde{u} \leq \tilde{v} \Leftrightarrow \max\{\tilde{u}, \tilde{v}\} = \tilde{v}$), we have to solve the following crisp LP

$$\begin{aligned} \max_{\mathbf{x}} \ \mathbf{c.x} \\ \text{subject to} \\ \mathbf{A}_i.\mathbf{x} \leq b_i \ (i \in \mathbb{N}_m) \\ \mathbf{A}_i.x - \underline{\mathbf{A}}_i \leq b_i - \underline{b}_i \ (i \in \mathbb{N}_m) \\ \mathbf{A}_i.x + \bar{\mathbf{A}}_i \leq b_i + \bar{b}_i \ (i \in \mathbb{N}_m) \\ \mathbf{x}, \lambda \geq 0. \end{aligned}$$

Case 3: LP problem with soft constraints . Let a maximizing problem with triangular coefficients (excluding the objective function) and soft constraint. We have

$$\begin{aligned} \max_{\mathbf{x}} \ \mathbf{c'.x}, \ (\mathbf{c}, \mathbf{x} \in \mathbb{R}^n) \\ \text{subject to} \\ \tilde{\mathbf{A}}_i.\mathbf{x} \lesssim \tilde{b}_i \ (i \in \mathbb{N}_m), \\ \mathbf{x} \geq 0. \end{aligned}$$

The entries $\tilde{a}_{ij},\ \tilde{b}_i$ are FNs of $\mathfrak{F}(\mathbb{R})$ whose values are known with imprecision. The fuzzy inequality $\precsim$ tells that the DM will allow some violation in the accomplishment of the constraint. The MFs $\mu_i : \mathfrak{F}(\mathbb{R}) \mapsto [0,1],\ i \in \mathbb{N}_m$ measure the adequacy between both sides of the constraint $\tilde{A}_i.\mathbf{x}$ and $\tilde{b}_i$. The fuzzy $\tilde{t}_i$ express the margins of tolerance for each constraint. Let ® be a ranking relation (<) between FNs, The auxiliary parametric LP problem is

$$\begin{aligned} &\max_{\mathbf{x}} \mathbf{c'.x},\ (\mathbf{c}, \mathbf{x} \in \mathbb{R}^n) \\ &\text{subject to} \\ \tilde{\mathbf{A}}_i.\mathbf{x}\ (<)\ &\tilde{b}_i + \tilde{t}_i(1-\alpha)\ (i \in \mathbb{N}_m), \\ &\mathbf{x} \geq 0,\ \alpha \in (0,1] \end{aligned}$$

The DM may choose different rules such as : $\tilde{x}\ (<)_1\ \tilde{y} \Leftrightarrow\ x \leq y$
or $\tilde{x}\ (<)_2\ \tilde{y} \Leftrightarrow\ \bar{x} \leq \underline{y}$. A different solution will be obtained for each rule.

Conversion of the fuzzy constraints into crisp inequalities. The FLP problem may be written

$$\begin{aligned} &\max_{\mathbf{x}} \mathbf{c'.x}\ (\mathbf{c}, \mathbf{x} \in \mathbb{R}^n) \\ &\text{subject to} \\ \tilde{\mathbf{A}}_i.\mathbf{x}\ \leq_{\Re} &\tilde{b}_i + \tilde{t}_i(1-\alpha),\ (i \in \mathbb{N}_m) \\ &\mathbf{x} \geq 0. \end{aligned}$$

In the constraint, the inequality rule $\leq_{\Re}$ is to be chosen by the DM among several ranking functions (or index) matching each FN into the real line. The DM may choose the rule 1: $\tilde{x}(<)_1\tilde{y} \Leftrightarrow x \leq y$ or the rule 2: $\tilde{x}(<)_2\tilde{y} \Leftrightarrow \bar{x} \leq \underline{y}$. Different solutions will be obtained.

Solving an auxiliary problem. Let a TFN be expressed by $\tilde{a} = (a, a^-, a^+)$, where $a^-,\ a^+$ are the lower and the upper limit of the support, respectively. Ranking the two fuzzy sides of the inequality may give the following auxiliary parametric LP problem

$$\begin{aligned} &\max_{\mathbf{x}} \mathbf{c'.x}\ (\mathbf{c}, \mathbf{x} \in \mathbb{R}^n) \\ &\text{subject to} \\ (A_i + A_i^- + A_i^+).\mathbf{x}\ \leq &(b_i + b_i^- + b_i^+) + (t_i + t_i^- + t_i^+)(1-\alpha),\ (i \in \mathbb{N}_m) \\ &\mathbf{x} \geq 0. \end{aligned}$$

2.4. Numerical example

This numerical example is due to (Delgado et al., 1990). The FLP problem is

$$\begin{aligned} &\max_{x_1,x_2} z = 5x_1 + 6x_2 \\ &\text{subject to} \\ &\tilde{3}x_1 + \tilde{4}x_2 \precsim \widetilde{18}, \\ &\tilde{2}x_1 + \tilde{1}x_2 \precsim \tilde{7}, \\ &x_1, x_2 \geq 0. \end{aligned}$$

The FNs take the form of tensors in Fig.8. A same FN as $\tilde{3}$ may thus have different definitions, in a given problem.

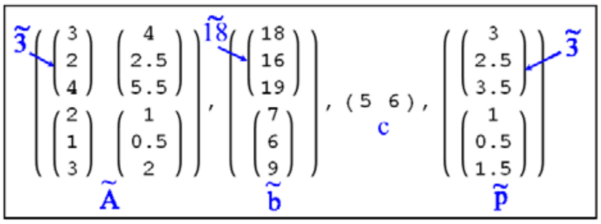

Figure8. Fuzzy parameters and tolerances

According to the ranking rule that the DM will choose, two different auxiliary problems and solutions are obtained. We have
rule 1 : $\tilde{x} \ <_{\Re_1} \ \tilde{y} \ \Leftrightarrow \ x \leq y$.

$$\begin{aligned} \max_{x_1,x_2} \ z &= 5x_1 + 6x_2 \\ &\text{subject to} \\ 3x_1 + 4x_2 &\leq 18 + 3(1-\alpha), \\ 2x_1 + x_2 &\leq 7 + (1-\alpha), \\ x_1, x_2 &\geq 0, \ \alpha \in (0,1] \end{aligned}$$

The parameterized solution with rule 1 given by *MATHEMATICA* is shown in Fig.9. The expression of the objective is $\frac{1}{5}(163 - 23\alpha)$, and that of the variables are $x_1 = \frac{1}{5}(11-\alpha)$ and $x_2 = \frac{3}{5}(6-\alpha)$. **rule 2** : $\tilde{x} \ <_{\Re_2} \ \tilde{y} \ \Leftrightarrow \ \bar{x} \leq \underline{y}$.

```
Maximize[Rationalize[{c.x,
    {a[[1, 1, 1]], a[[1, 2, 1]]}.x ≤ b[[1, 1, 1]] + t[[1, 1, 1]] (1 - α) &&
     {a[[2, 1, 1]], a[[2, 2, 1]]}.x ≤ b[[1, 2, 1]] + t[[1, 2, 1]] (1 - α) &&
     x1 ≥ 0 && x2 ≥ 0 && 0 < α ≤ 1}],
  x] // Simplify

{{ 1/5 (163 - 23 α)    0 < α ≤ 1
   -∞                  True        ,
 {x1 → Indeterminate   α > 1 || α ≤ 0  , x2 → Indeterminate   α > 1 || α ≤ 0 }}
       (11-α)/5        True                   -3/5 (-6 + α)   True
```

Figure9. Parameterized solution with rule 1

$$\begin{aligned} \max_{x_1,x_2} \ z &= 5x_1 + 6x_2 \\ &\text{subject to} \\ 4x_1 + 5.5x_2 &\leq 16 + 2.5(1-\alpha), \\ 3x_1 + 2x_2 &\leq 6 + .5(1-\alpha), \\ x_1, x_2 &\geq 0, \ \alpha \in (0,1] \end{aligned}$$

The parameterized solution with rule 2 given by *MATHEMATICA* is shown Fig.10. The solutions are also expressions defined on intervals. The optimal objective is piecewise with $\frac{3}{2}(13-\alpha)$ for $\alpha \in (0, \frac{5}{9}]$ and $\frac{1}{34}(683 - 87\alpha)$ for $\alpha \in (\frac{5}{9}, 1]$. The solutions for x_1 and x_2 are shown in Fig.11.

```
Maximize[
 Rationalize[{5 x1 + 6 x2, 4 x1 + 5.5 x2 ≤ 16 + 2.5 (1 - α),
   3 x1 + 2 x2 ≤ 6 + .5 (1 - α) , x1 ≥ 0, x2 ≥ 0, 0 < α ≤ 1}], {x1, x2}]
```

$$\left\{\begin{cases} \frac{1}{34}(683-87\alpha) & \frac{5}{9}<\alpha\le 1 \\ \frac{1}{2}(39-3\alpha) & 0<\alpha\le\frac{5}{9}, \\ -\infty & \text{True} \end{cases}\right.$$

$$\left\{x1\to\begin{cases} 0 & 0<\alpha\le\frac{5}{9} \\ \frac{1}{7}\left(-222+30\alpha-\frac{11}{34}(-683+87\alpha)\right) & \frac{5}{9}<\alpha\le 1, \\ \text{Indeterminate} & \text{True} \end{cases}\right.$$

$$x2\to\begin{cases} \frac{1}{12}(39-3\alpha) & 0<\alpha\le\frac{5}{9} \\ \frac{1}{6}\left(\frac{1}{34}(683-87\alpha)-\frac{5}{7}\left(-222+30\alpha-\frac{11}{34}(-683+87\alpha)\right)\right) & \frac{5}{9}<\alpha\le 1 \\ \text{Indeterminate} & \text{True} \end{cases}\Bigg\}\Bigg\}$$

Figure10. Parameterized solution with rule 2

$$x_1^* = \begin{cases} \frac{1}{34}(-5+9\alpha),\ x\in(\frac{5}{9},1] \\ 0,\ x\in(0,\frac{5}{9}] \end{cases} \qquad x_2^* = \begin{cases} \frac{1}{17}(59-11\alpha),\ x\in(\frac{5}{9},1] \\ \frac{1}{4}(13-\alpha),\ x\in(0,\frac{5}{9}] \end{cases}$$

Figure11. Optimal strategies x_1^* and x_2^* with rule 2

3. Single objective fuzzy matrix games

3.1. Problem formulation and equilibrium solution

The problem formulation and the equilibrium solutions of bi-matrix games are compared for both crisp and fuzzy versions.

Problem formulation

1) Crisp bi-matrix game formulation. A two-person bi-matrix game is represented by $G = (S^m, S^n, \mathbf{A}, \mathbf{B})$, where S^m, S^n are the strategy spaces of the Players I and II, respectively. The two Players I and II have mixed strategies which are the m-dimensional vector $\mathbf{x}$ and the n-dimensional vector $\mathbf{y}$, respectively. The strategy spaces are defined by the convex polytopes

$$S^m = \{\mathbf{x} \in \mathbb{R}_+^m,\ \mathbf{x}'\mathbf{e}_m = 1\}$$

and

$$S^n = \{\mathbf{y} \in \mathbb{R}_+^n,\ \mathbf{y}'\mathbf{e}_n = 1\}.$$

The payoffs of Players I and II are $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ with real entries, respectively. The payoff domains of Players I and II are the sets $D_1 = \{\mathbf{x'Ay} |\ \mathbf{x} \in S^m\} \subseteq \mathbb{R}$ and $D_2 = \{\mathbf{x'By} |\ \mathbf{y} \in S^n\} \subseteq \mathbb{R}$.
The programming problems of the Players I and II are

$$\{\max_{\mathbf{x}}\ \mathbf{x'Ay} \text{ subject to } \mathbf{e'}_m\mathbf{x} = 1,\ \mathbf{x} \geq 0\}$$

and

$$\{\max_{\mathbf{y}} \ \mathbf{x'By} \text{ subject to } \mathbf{e}'_n\mathbf{y} = 1, \ \mathbf{y} \geq 0\},$$

respectively.

2) Fuzzy bi-matrix game formulation. A (not completely) fuzzified two-person bi-matrix game with fuzzy goals and fuzzy payoffs, is represented by

$$G = (S^m, S^n, \widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}, \tilde{v}, \tilde{p}, \tilde{p}', \tilde{w}, \tilde{q}, \tilde{q}', \lesssim, \gtrsim),$$

where $\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}$ are the payoffs $m \times n$ matrices with fuzzy entries, $\tilde{v}, \tilde{w}$ the aspiration levels of Players I and II, $\tilde{p}, \tilde{p}'$ the fuzzy tolerance levels for Player I, $\tilde{q}, \tilde{q}'$ the fuzzy tolerance levels for Player II and $\lesssim, \gtrsim$ the fuzzy inequalities. A fuzzy goal for Player I is a fuzzy set $\tilde{G}_1$ which MF is $\mu_1 : D_1 \mapsto [0, 1]$. The fuzzy goal of Player II is similarly defined. The fuzzy payoff matrix $\widetilde{\mathbf{A}}$ may have one LR-representation for the entries such as $\tilde{a}_{ij} = (a_{ij}, \delta^-_{a_{ij}}, \delta^+_{a_{ij}})_{LR}$, where a_{ij} denotes the mean value, $\delta^-_{a_{ij}}$ and $\delta^+_{a_{ij}}$ the left and right spreads.

Equilibrium solution

1) Crisp bi-matrix game solution. The value of the game is obtained at the point $(\mathbf{x}'^*\mathbf{A}\mathbf{y}^*, \mathbf{x}'^*\mathbf{B}\mathbf{y}^*)$. According to the Equivalence Theorem, the conditions for the pair $(\mathbf{x}^*, \mathbf{y}^*)$ to be an equilibrium point is the solution of the QP problem

$$\begin{aligned} \max_{\mathbf{x},\mathbf{y},p,q} \ \mathbf{x'}(\mathbf{A}+\mathbf{B})\mathbf{y} - p - q \\ \text{subject to} \\ \mathbf{B}'\mathbf{x} \leq q\mathbf{e}_n, \\ \mathbf{A}\mathbf{y} \leq p\mathbf{e}_m, \\ \mathbf{x}'\mathbf{e}_m = 1, \\ \mathbf{y}'\mathbf{e}_n = 1, \\ \mathbf{x} \geq 0, \ \mathbf{y} \geq 0, \end{aligned}$$

where $p, q \in \mathbb{R}$ are the negative of the multipliers associated with the constraints.

2) Fuzzy bi-matrix game solution

Definition 4. (Bellman-Zadeh decision principle). Based on the principle of decision by Bellman-Zadeh, the fuzzy decision is expressed as the intersection of the fuzzy goals and expected payoffs, such as for Player I,

$$\mu_{a(\mathbf{x},\mathbf{y})} = \min\left\{\mu_{\mathbf{x}\tilde{\mathbf{A}}\mathbf{y}}(p), \mu_{\tilde{G}_1}(p)\right\}.$$

The fuzzy decision for Player II, is similarly defined.

Definition 5. (Degree of attainment of the fuzzy goal). A degree of attainment of the fuzzy goal is defined as the maximum of the MF $\mu_{a(\mathbf{x},\mathbf{y})}$. We have

$$d_1(\mathbf{x}, \mathbf{y}) = \max_p\left(\min\left\{\mu_{\mathbf{x}\tilde{\mathbf{A}}\mathbf{y}}(p), \mu_{\tilde{G}_1}(p)\right\}\right).$$

The degree of attainment of the fuzzy goal for Player II $d_2(\mathbf{x},\mathbf{y})$ is similarly defined. According to the Nishizaki and Sakawa's model, each player is supposed to maximize the degree of attainment of his goal. An equilibrium solution is then defined w.r.t. the degree of attainment of the fuzzy goals by the two players. Let $G = (S^m, S^n, \widetilde{\mathbf{A}}, \widetilde{\mathbf{B}})$ be a fuzzy bi-matrix game, the Nash equilibrium solution w.r.t. the degree of attainment of the fuzzy goal is a pair of strategies $(\mathbf{x}^*,\mathbf{y}^*)$ if, for all other strategies, we have

$$\begin{aligned} d_1(\mathbf{x}^*,\mathbf{y}^*) &\geq d_1(\mathbf{x},\mathbf{y}^*) \text{ for all } \mathbf{x} \in S^m, \\ d_2(\mathbf{x}^*,\mathbf{y}^*) &\geq d_2(\mathbf{x}^*,\mathbf{y}) \text{ for all } \mathbf{y} \in S^n. \end{aligned}$$

The programming problem of the Player I is

$$\begin{aligned} \max_{\mathbf{x}}\ d_1(\mathbf{x},\mathbf{y}^*) = \frac{\mathbf{x}'(\mathbf{A}+\Delta_{\mathbf{A}})\mathbf{y}^* - \underline{a}}{\bar{a} - \underline{a} + \mathbf{x}'\Delta_{\mathbf{A}}\mathbf{y}^*}& \\ \text{subject to}& \\ \mathbf{x}'\mathbf{e}_m = 1,& \\ \mathbf{x} \geq 0.& \end{aligned}$$

The programming problem of the Player II is similarly defined. Applying the K-T conditions, we have the equivalence Theorem.

3.2. Bi-matrix games with fuzzy payoffs

The problems of Player I and Player II are solved according to identical steps. The initial problem is transformed into an another problem, by changing the variables. A ranking function is then introduced. Finally, the resulting auxiliary problem is solved. A numerical example illustrates the procedure.

Problem of the Player I. The fuzzy matrix game problem of the maximizing Player I in a fuzzy environment is

$$\begin{aligned} \max\ v& \\ \text{subject to}& \\ \sum_{i=1}^{m} \tilde{a}_{ij}x_i \gtrsim\ v,\ (j \in \mathbb{N}_n)& \\ \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0\ (i \in \mathbb{N}_m).& \end{aligned}$$

The Player I's payoffs $\tilde{a}_{ij}$ are fuzzy. The fuzzy inequality $\precsim$ tells that the DM will allow some violation in the accomplishment of the constraint.

1) Variable changing. The variables are changed into $u_i = x_i/v$ $(i \in \mathbb{N}_m)$. We have $\sum_{i=1}^{m} u_i = 1/v$, then $v = 1/\sum_{i=1}^{m} u_i$. The initial problem is transformed into

$$\begin{aligned} \min\ \sum_{i=1}^{m} u_i& \\ \text{subject to}& \\ \sum_{i=1}^{m} \tilde{a}_{ij}u_i \gtrsim\ 1,\ (j \in \mathbb{N}_n)& \\ u_i \geq 0\ (i \in \mathbb{N}_m).& \end{aligned}$$

2) Introduction of a ranking function. For solving the FLP problem in canonical form, a ranking function is introduced to compare both fuzzy sides of the inequality. The Player I's problem is transformed into the parametric LP problem

$$\min \sum_{i=1}^{m} u_i$$
$$\text{subject to}$$
$$\sum_{i=1}^{m} \tilde{a}_{ij} u_i \geq_{\Re} 1 - \tilde{p}_j(1-\alpha), \ (j \in \mathbb{N}_n)$$
$$u_i \geq 0 \ (i \in \mathbb{N}_m), \ \alpha \in (0,1].$$

The fuzzy $\tilde{p}_i$'s are the maximum violation that the Player I will allow for the constraints.

3) Solution of the auxiliary problem. Let the Player I's payoffs $\tilde{a}_{ij}$ be TFNs expressed by $\tilde{a}_{ij} = (a_{ij}, a_{ij}^{-}, a_{ij}^{+})$, where a_{ij}^{-}, a_{ij}^{+} are the lower and the upper limit of the support, respectively. Ranking the two sides of the inequalities leads to the following auxiliary parametric LP problem

$$\min \sum_{i=1}^{m} u_i$$
$$\text{subject to}$$
$$\sum_{i=1}^{m} (a_{ij} + a_{ij}^{-} + a_{ij}^{+}) u_i \ \geq 3 + (p_i + p_i^{-} + p_i^{+})(1-\alpha), \ (j \in \mathbb{N}_n)$$
$$u_i \geq 0 \ (i \in \mathbb{N}_m), \ \alpha \in (0,1].$$

4) Numerical example. This numerical example is due to (Campos,1989). The fuzzy payoff matrix of Player I is

$$\tilde{A} = \begin{pmatrix} \widetilde{180} & \widetilde{156} \\ \widetilde{90} & \widetilde{180} \end{pmatrix}$$

The TFNs are defined by $\widetilde{180} = (180, 175, 190)$, $\widetilde{156} = (156, 150, 158)$, $\widetilde{90} = (90, 80, 100)$. The fuzzy margins are $\tilde{p}_1 = \tilde{p}_2 = (0.10, 0.08, 0.11)$ for the Player I. The FLP problem is

$$\min \ u_1 + u_2$$
$$\text{subject to}$$
$$\widetilde{180} u_1 + \widetilde{90} u_2 \geq_{\Re} 1 - \widetilde{0.10}(1-\alpha)$$
$$\widetilde{90} u_1 + \widetilde{180} u_2 \geq_{\Re} 1 - \widetilde{0.10}(1-\alpha)$$
$$u_1, \ u_2 \geq 0, \ \alpha \in (0,1].$$

The auxiliary problem is

$$\min \ u_1 + u_2$$
$$\text{subject to}$$
$$545 u_1 + 270 u_2 \geq 3 - 0.29(1-\alpha)$$
$$464 u_1 + 545 u_2 \geq 3 - 0.29(1-\alpha)$$
$$u_1, \ u_2 \geq 0, \ \alpha \in (0,1].$$

Solving the auxiliary problem and changing the variables, the optimal Player I's strategies are $x^* = (0.77, 0.23)$ and $v(\alpha) = \frac{482.43}{3-0.29(1-\alpha)}$, $\alpha \in (0, 1]$.

Problem of the Player II. The fuzzy matrix game problem of the minimizing Player II in a fuzzy environment is

$$\begin{aligned} &\min\ w \\ &\text{subject to} \\ &\sum_{j=1}^{n} \tilde{a}_{ij} y_j \gtrsim\ w,\ (i \in \mathbb{N}_m) \\ &\sum_{j=1}^{n} y_j = 1,\ y_j \geq 0\ (j \in \mathbb{N}_n). \end{aligned}$$

The losses of Player II $\tilde{a}_{ij}$ are fuzzy numbers of $\mathfrak{F}(\mathbb{R})$ whose values are known with imprecision.

1) Variable changing. Let change the variables into $s_j = y_j/w$ $(j \in \mathbb{N}_n)$. We have $\sum_{j=1}^{n} s_j = 1/w$, the $w = 1/\sum_{j=1}^{n} s_j$. The initial problem is transformed into

$$\begin{aligned} &\max \sum_{j=1}^{n} s_i \\ &\text{subject to} \\ &\sum_{j=1}^{n} \tilde{a}_{ij} s_j \lesssim\ 1,\ (i \in \mathbb{N}_m) \\ &s_j \geq 0\ (j \in \mathbb{N}_n). \end{aligned}$$

The RHS of the fuzzy inequality is transformed to a crisp number.

2) Introduction of a ranking function. For solving the FLP problem in canonical form, we apply the following procedure : a ranking function is introduced to compare both fuzzy sides of the inequality, and solving a parametric LP problem. The problem of the player II is transformed into

$$\begin{aligned} &\max \sum_{j=1}^{m} s_i \\ &\text{subject to} \\ &\sum_{j=1}^{n} \tilde{a}_{ij} s_j \leq_{\Re}\ 1 + \tilde{q}_i(1-\alpha),\ (i \in \mathbb{N}_m) \\ &s_j \geq 0\ (j \in \mathbb{N}_n),\ \alpha \in (0, 1]. \end{aligned}$$

The fuzzy $\tilde{q}_j$'s are the maximum violation that the Player II will allow for the constraints.

3) Solution of the auxiliary problem . Let the Player II's losses $\tilde{a}_{ij}$ be TFNs be expressed by $\tilde{a}_{ij} = (a_{ij}, a_{ij}^-, a_{ij}^+)$. Ranking the two fuzzy sides of the inequality produces the following auxiliary parametric LP problem

$$\max \sum_{j=1}^{n} s_i$$
$$\text{subject to}$$
$$\sum_{j=1}^{n}(a_{ij}+a_{ij}^{-}+a_{ij}^{+})s_j \ \leq 3+(q_i+q_i^{-}+q_i^{+})(1-\alpha),\ (i\in\mathbb{N}_m)$$
$$s_j\geq 0\ (j\in\mathbb{N}_n),\ \alpha\in(0,1].$$

4) Numerical example. The fuzzy losses matrix of Player II is

$$\tilde{A}=\begin{pmatrix}\widetilde{180} & \widetilde{156}\\ \widetilde{90} & \widetilde{180}\end{pmatrix}.$$

The TFNs are defined by $\widetilde{180}=(180,175,190)$, $\widetilde{156}=(156,150,158)$, $\widetilde{90}=(90,80,100)$. The fuzzy margins are $\tilde{q}_1=\tilde{q}_2=(0.15,0.14,0.17)$ for the Player II.

The FLP problem is

$$\max\ s_1+s_2$$
$$\text{subject to}$$
$$\widetilde{180}s_1+\widetilde{156}s_2\leq_{\Re} 1-\widetilde{0.15}(1-\alpha)$$
$$\widetilde{90}s_1+\widetilde{180}s_2\leq_{\Re} 1-\widetilde{0.15}(1-\alpha)$$
$$s_1,\ s_2\geq 0,\ \alpha\in(0,1].$$

The auxiliary problem is

$$\max\ s_1+s_2$$
$$\text{subject to}$$
$$545s_1+464s_2\leq 3+0.46(1-\alpha)$$
$$270s_1+545s_2\leq 3+0.46(1-\alpha)$$
$$s_1,\ s_2\geq 0,\ \alpha\in(0,1].$$

Solving the auxiliary problem and changing the variables, the optimal Player's II strategies are $y^*=(0.23,0.77)$ and $w(\alpha)=\frac{482.43}{3+0.49(1-\alpha)},\ \alpha\in(0,1]$.

3.3. Bi-matrix games with Fuzzy goal

The first task of each player is to determine a linear MF $\mu(x,y)$ for each pair of strategies (x,y). We assume that the degree of satisfaction of each player depends on the expected payoff $\mathbf{xAy}$. For Player I, the maximin solution w.r.t. a degree of achievement of the fuzzy goal is an optimal solution. Similarly for Player II, the minimax solution w.r.t. a degree of achievement of the fuzzy goal is an optimal solution. Let the single-objective matrix game $G=(S^m,S^n,\mathbf{A})$ with fuzzy goals where S^m and S^n denote the compact convex strategy spaces of Players, such that $S^m=\{\mathbf{x}\in\mathbb{R}_+^m,\ \mathbf{e}'\mathbf{x}=1\}$ and $S^n=\{\mathbf{y}\in\mathbb{R}_+^n,\ \mathbf{e}'\mathbf{y}=1\}$, and where $\mathbf{A}\in\mathbb{R}^{m\times n}$ is the payoff matrix of the game with real entries.

Maximin problem of the Player I. For any pair of strategies (x, y) the MF $\mu(x, y)$ depends on the expected payoff $\mathbf{xAy}$. Assume that the degree of satisfaction increases linearly, we have

$$\mu(\mathbf{xAy}) = \begin{cases} 1, \quad \mathbf{xAy} \geq \bar{a} \\ \frac{\mathbf{xAy} - \underline{a}}{\bar{a} - \underline{a}}, \ \underline{a} \leq \mathbf{xAy} \leq \bar{a} \\ 0, \ \mathbf{xAy} \leq \underline{a}, \end{cases}$$

where $\bar{a}$ and $\underline{a}$ are the best and the worst degree of satisfaction to the Player I, respectively. These extremal values are determined by

$$\bar{a} = \max_x \max_y \mathbf{xAy} = \max_{i \in \mathbb{N}_m} \max_{j \in \mathbb{N}_n} a_{ij}$$
$$\underline{a} = \min_x \min_y \mathbf{xAy} = \min_{i \in \mathbb{N}_m} \min_{j \in \mathbb{N}_n} a_{ij}$$

The MF of the fuzzy goal is shown in Fig.12.

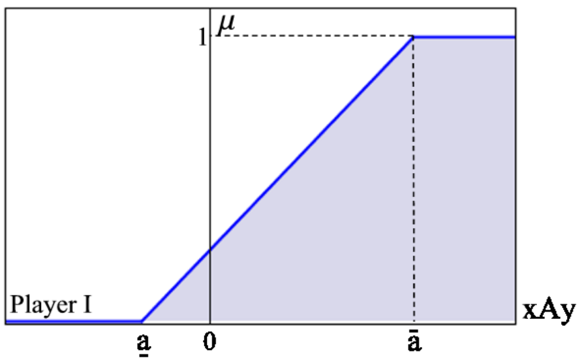

Figure12. Fuzzy goal membership function of Player I

The maximin solution of Player I is given by the following Theorem.

Theorem 4. *(Maximin solution). For a single-objective two-person matrix game with a linearly fuzzy goal function, the Player I's maximin solution w.r.t. a degree of achievement of the fuzzy goal is equal to an optimal solution to the LP problem*

$$\begin{aligned} &\max \ \lambda \\ &\textit{subject to} \\ &\frac{1}{\bar{a} - \underline{a}} \left(\sum_{i=1}^{m} a_{ij} x_i - \underline{a} \right) \geq \lambda, (j \in \mathbb{N}_n) \\ &\boldsymbol{e}'\boldsymbol{x} = 1, \ (\boldsymbol{e}, \ \boldsymbol{x} \in \mathbb{R}^m), \\ &\boldsymbol{x} \geq 0. \end{aligned}$$

`Proof`: see (Nishizaki and Sakawa, 2001), page 39. □

In the Cook's example (Cook, 1976), a 3×3 payoff matrix is given by

$$A = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{pmatrix}$$

We have $\bar{a} = 6$ and $\underline{a} = -2$. Then, we have to solve the LP problem

$$
\begin{aligned}
\max_{\mathbf{x},\lambda}\ & \lambda \\
\text{subject to} & \\
2x_1 - x_2 + 2 &\geq 8\lambda, \\
5x_1 - 2x_2 + 3x_3 + 2 &\geq 8\lambda, \\
x_1 + 6x_2 - x_3 + 2 &\geq 8\lambda, \\
\mathbf{e}.\mathbf{x} = 1,\ \mathbf{x} &\geq 0.
\end{aligned}
$$

The Player I's optimal strategies (in Fig.13) are $x_1^* = .875, x_2^* = .125$ and $x_3^* = 0$.

```
NMaximize[{λ,
   (B[[1]].x - aw) / (ab - aw) ≥ λ,
   (B[[2]].x - aw) / (ab - aw) ≥ λ,
   (B[[3]].x - aw) / (ab - aw) ≥ λ,
   x1 + x2 + x3 == 1, x1 ≥ 0, x2 ≥ 0, x3 ≥ 0},
  {x1, x2, x3, λ}]
{0.453125, {x1 → 0.875, x2 → 0.125, x3 → 0., λ → 0.453125}}
```

Figure13. Player I's solution

Minimax problem of the Player II. For any pair of strategies (x, y) the MF $\mu(x, y)$ depends on the expected payoff $\mathbf{xAy}$. Assume that the degree of satisfaction decreases linearly, we have

$$
\mu(\mathbf{xAy}) = \begin{cases} 1, & \mathbf{xAy} \leq \underline{a} \\ \frac{\bar{a} - \mathbf{xAy}}{\bar{a} - \underline{a}}, & \underline{a} \leq \mathbf{xAy} \leq \bar{a} \\ 0, & \mathbf{xAy} \geq \bar{a}, \end{cases}
$$

where $\bar{a}$ and $\underline{a}$ are the worst and the best degree of satisfaction to the Player II, respectively. These extremal values are determined by

$$
\begin{aligned}
\bar{a} &= \max_{x} \max_{y} \mathbf{xAy} = \max_{i \in \mathbb{N}_m} \max_{j \in \mathbb{N}_n} a_{ij} \\
\underline{a} &= \min_{x} \min_{y} \mathbf{xAy} = \min_{i \in \mathbb{N}_m} \min_{j \in \mathbb{N}_n} a_{ij}
\end{aligned}
$$

The MF of the fuzzy goal is shown in Fig.14.

The minimax solution of Player II is given by the following theorem.

Theorem 5. *(Minimax solution). For a single-objective two-person matrix game with a linearly fuzzy goal function, the Player II's minimax solution w.r.t. a degree*

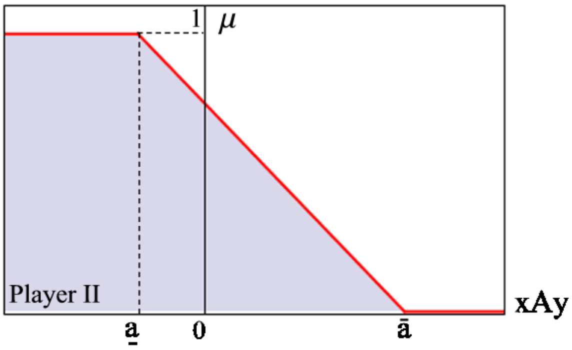

Figure14. Fuzzy goal membership function of Player II

of achievement of the fuzzy goal is equal to an optimal solution to the LP problem

$$\begin{aligned} &\min\ \lambda \\ &\textit{subject to} \\ \frac{1}{\bar{a}-\underline{a}}\left(\sum_{j=1}^{n} a_{ij}y_i - \underline{a}\right) &\leq \lambda + 1, (i \in \mathbb{N}_m) \\ \boldsymbol{e}'\boldsymbol{y} &= 1,\ (\boldsymbol{e},\ \boldsymbol{y} \in \mathbb{R}^n), \\ \boldsymbol{y} &\geq 0. \end{aligned}$$

`Proof`: see (Nishizaki and Sakawa, 2001), page 41. □

In the Cook's example Cook, 1976, a 3×3 payoff matrix is given by

$$A = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{pmatrix}$$

We have $\bar{a} = 6$ and $\underline{a} = -2$. Then we have to solve the LP problem

$$\begin{aligned} &\min_{\mathbf{y},\lambda}\ \lambda \\ &\text{subject to} \\ 2y_1 + 5y_2 + y_3 + 2 &\leq 8(1+\lambda), \\ -y_1 - 2y_2 + 6x_3 + 2 &\leq 8(1+\lambda), \\ 3y_2 - y_3 + 2 &\leq 8(1+\lambda), \\ \mathbf{e}.\mathbf{y} = 1,\ \mathbf{y} &\geq 0. \end{aligned}$$

The Player II's optimal strategies, in Fig.15, are

$$y_1^* = .625, y_2^* = 0, y_3^* = 0.375$$

```
NMinimize[{λ,
  (A[[1]].y - aw) / (ab - aw) ≤ λ + 1, (A[[2]].y - aw) / (ab - aw) ≤ λ + 1,
  (A[[3]].y - aw) / (ab - aw) ≤ λ + 1,
  y1 + y2 + y3 == 1, y1 ≥ 0, y2 ≥ 0, y3 ≥ 0}, {y1, y2, y3, λ}]
{-0.546875, {y1 → 0.625, y2 → 0., y3 → 0.375, λ → -0.546875}}
```

Figure15. Player II's solution

Appendix

A Fuzzy number arithmetic

Two methods can be used in fuzzy arithmetics: one method is based on interval arithmetics and the other uses the extension principle (Mareš, 1994 - Nguyen and Walker, 2006).

A1. Interval arithmetics

Interval arithmetics are based on two properties of the FNs [3]: 1) each FN is uniquely represented by its α-cuts and 2) the α-cuts are closed intervals of real numbers for all $\alpha \in (0,1]$. Let $\tilde{A}$ and $\tilde{B}$ denote TFNs and let $*$ be one of the four arithmetic operations: addition, substraction, multiplication and division. A fuzzy set $\tilde{A} * \tilde{B}$ is defined by the α-cuts

$$^{\alpha}(\mathbf{A} * \mathbf{B}) =^{\alpha} \mathbf{A} *^{\alpha} \mathbf{B}, \text{ for any } \alpha \in (0,1].$$

Theorem 6. *(First Decomposition Theorem). For every fuzzy set $\tilde{A} \in \mathfrak{F}(\mathbb{R})$, we have*

$$\tilde{A} = \bigcup_{\alpha \in [0,1]} {}_{\alpha}\tilde{A},$$

where the α-cuts are converted into the fuzzy set ${}_{\alpha}\tilde{A}$, defined as

$$\tilde{A} = \alpha\ {}^{\alpha}A.$$

`Proof`: see (Klir and Yuan, 1995, pages 41-42).□
The numerical example in (Klir and Yuan, 1995, page 105) considers two TFNs $\tilde{A}$ and $\tilde{B}$ defined in Fig.16.

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x+1}{2}, & x \in (-1,1] \\ \frac{3-x}{2}, & x \in (1,3] \\ 0, & \text{elsewhere} \end{cases} \quad \textbf{and} \quad \mu_{\tilde{B}}(x) = \begin{cases} \frac{x-1}{2}, & x \in (1,3] \\ \frac{5-x}{2}, & x \in (3,5] \\ 0, & \text{elsewhere} \end{cases}$$

Figure16. Fuzzy numbers $\tilde{A}$ and $\tilde{B}$

[3] This appendix is inspired from (Klir and Yuan, 1995). The computations and plots use the software *MATHEMATICA 7.0.1*.

The α-cuts are

$$^{\alpha}A = [2\alpha - 1, 3 - 2\alpha] \text{ and } ^{\alpha}B = [2\alpha + 1, 5 - 2\alpha].$$

The four arithmetic operations on closed intervals are defined in Fig.17 [4]

Addition $^{\alpha}(A+B) = [4\alpha, 8-4\alpha],\ \alpha \in (0,1]$	**Substraction** $^{\alpha}(A-B) = [4\alpha-6, 2-4\alpha],\ \alpha \in (0,1]$
Multiplication $^{\alpha}(A.B) = \begin{cases} [-4\alpha^2+12\alpha-5, 4\alpha^2-16\alpha+15],\ \alpha \in (0,.5] \\ [-4\alpha^2-1, 4\alpha^2-16\alpha+15],\ \alpha \in (.5,1] \end{cases}$	**Division** $^{\alpha}(A/B) = \begin{cases} [\frac{2\alpha-1}{2\alpha+1}, \frac{3-2\alpha}{2\alpha+1}],\ \alpha \in (0,.5] \\ [\frac{3-2\alpha}{5-2\alpha}, \frac{3-2\alpha}{2\alpha+1}],,\ \alpha \in (.5,1] \end{cases}$

Figure17. α-cuts with the arithmetic operations

The resulting FNs are represented in Figs.1819.

Addition $\mu_{\tilde{A}+\tilde{B}}(x) = \begin{cases} \frac{x}{4},\ x \in (0,4] \\ \frac{8-x}{4},\ x \in (4,8] \\ 0,\ \text{elsewhere} \end{cases}$	**Substraction** $\mu_{\tilde{A}-\tilde{B}}(x) = \begin{cases} \frac{x+6}{4},\ x \in (-6,-2] \\ \frac{2-x}{4},\ x \in (-2,2] \\ 0,\ \text{elsewhere} \end{cases}$
Multiplication $\mu_{\tilde{A}.\tilde{B}}(x) = \begin{cases} \frac{1}{2}(3-\sqrt{4-x}),\ x \in (-5,0] \\ \frac{1}{2}\sqrt{1+x},\ x \in (0,3] \\ \frac{1}{2}(4-\sqrt{1+x}),\ x \in (3,15] \\ 0,\ \text{elsewhere} \end{cases}$	**Division** $\mu_{\tilde{A}/\tilde{B}}(x) = \begin{cases} \frac{x+1}{2-2x},\ x \in (-1,0] \\ \frac{5x+1}{2x+2},\ x \in (0,\frac{1}{3}] \\ \frac{3-x}{2x+2},\ x \in (\frac{1}{3},3] \\ 0,\ \text{elsewhere} \end{cases}$

Figure18. Fuzzy arithmetics

A2. Extension principle

The extension principle supposes that the standard arithmetic operations on real numbers are extended to FNs.

Theorem 7. *(Extension principle). Let $*$ denote one of the four operations (addition +, substraction -, multiplication ., and division /), and let $\tilde{A}, \tilde{B}$ denote FNs. We define a continuous FN $\tilde{A} * \tilde{B}$ on $\mathbb{R}$ by*

$$\mu_(\tilde{A} * \tilde{B})(z) = \sup_{z=x*y} \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\},\ for\ all\ z \in \mathbb{R}.$$

[4] For the multiplication, we have

$$[a,b].[d,e] = [\min\{ad, ae, bd, be\}, \max\{ad, ae, bd, be\}].$$

For the division

$$[a,b]/[d,e] = [a,b].[1/e, 1/d],$$

we use the same rule, provided that 0 is not in $[d,e]$.

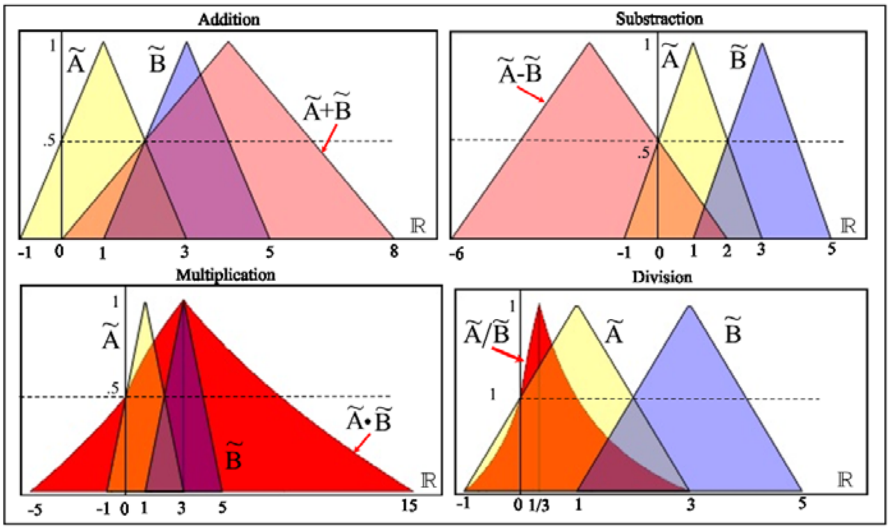

Figure19. Arithmetics with triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$

`Proof`: see (Klir and Yuan, 1995, pages 106-109).□

B Function principle

The Chen's function principle is more appropriate for multiple (≥ 3) trapezoidal FNs rather than the extension principle. Let two trapezoidal FNs $\tilde{A}$ and $\tilde{B}$ be $\tilde{A} = (a_1, a_2, a_3, a_4; w_1)$ and $\tilde{B} = (b_1, b_2, b_3, b_4; w_2)$, where w_1 and w_2 denote the height of $\tilde{A}$ and $\tilde{B}$, respectively. The two FNs are illustrated in Fig.20.

We have to calculate $\tilde{C} = \tilde{A} * \tilde{B}$, where $* \in \{+, -, ., /\}$. The resulting fuzzy trapezoidal number is defined by

$$C = (c_1, c_2, c_3, c_4; w),$$

where $w = \min\{w_1, w_2\}$. Defining the set

$$T = \{a_1 * b_1, a_1 * b_1, a_1 * b_4, a_4 * b_1, a_4 * b_4\},$$

we deduce $c_1 = \min T$ and $c_4 = \max T$. Letting

$$s_1 = \min\{x, \mu_{\tilde{A}}(x) \geq w\},\ \ s_2 = \min\{x, \mu_{\tilde{B}}(x) \geq w\},$$

$$t_1 = \max\{x, \mu_{\tilde{A}}(x) \geq w\},\ \ t_2 = \max\{x, \mu_{\tilde{B}}(x) \geq w\},$$

we define the set

$$T_1 = \{s_1 * s_2, s_1 * t_2, t_1 * s_2, t_1 * t_2\},$$

and deduce

$$c_2 = \min\{T_1\}\ and\ c_3 = \max\{T_1\}.$$

The results of the fuzzy arithmetic, using the function principle are shown in Fig.21.

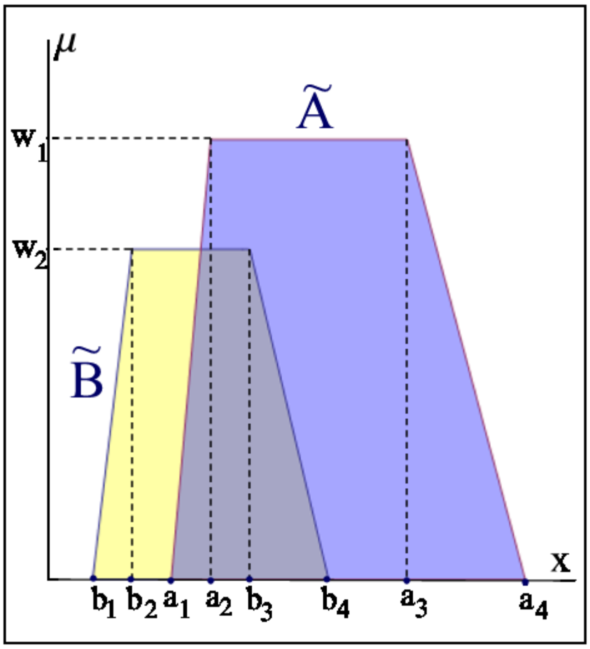

Figure20. Trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$

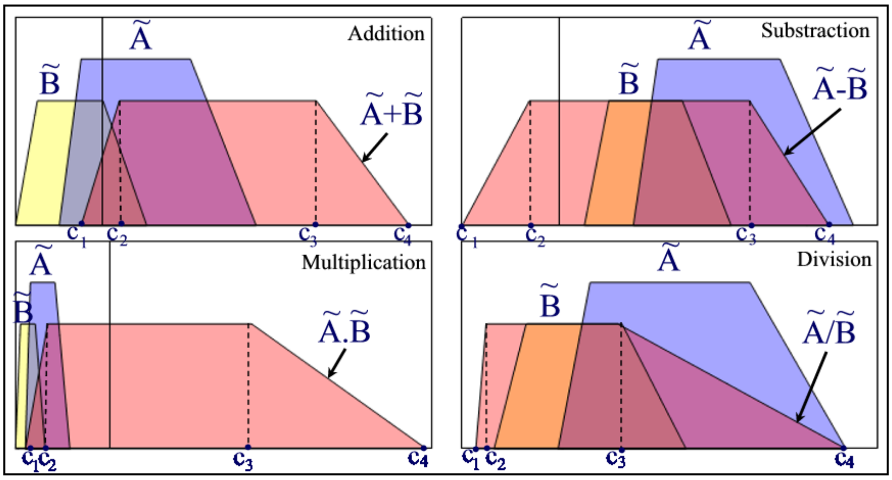

Figure21. Arithmetics with trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$

Abbreviation	Definition	Symbol	Definition
DM	decision maker	$\tilde{A}$	fuzzy number A
FLG	fuzzy logic game	(A, A^-, A^+)	(mean, lower, upper limit)
FLP	fuzzy linear programming	${}^{\alpha}A$	α-cut of A
FN	fuzzy number	${}_{\alpha}\tilde{A}$	special fuzzy set
K-T	Kuhn-Tucker conditions	e_n	n-dimensional vector of ones
LP	linear programming	$\tilde{p}$	fuzzy tolerance
LR-type	left-right representation	δ^-, δ^+	left, right spread
MF	membership function	$\mu_{\tilde{A}}(x)$	left, right spread
MOT	management of technology	$\mathfrak{F}(\mathbb{R})$	fuzzy sets family
QP	quadratic programming	$\mathbb{N}_n$	n positive integers
RHS	right-hand side	$\mathbb{R}$	real line
R&D	research and development	®	ranking relation
TFN	triangular fuzzy number	$(<)_{\mathcal{R}}$	inequality rule $\mathcal{R}$
		$\lesssim, \gtrsim$	fuzzy inequality
		$\bigwedge$	Min operator
		$\oplus,\ \ominus,\ \otimes\ \oslash$	fuzzy operations

Table1. Index of abbreviations and symbols

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