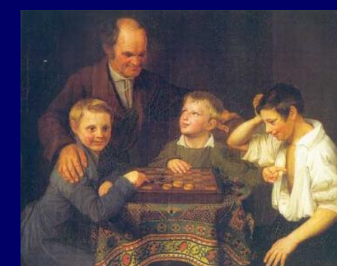
Fuzzy Logic Games in Economics



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Introduction

- A. Main references
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A. Main references

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B. Research's history on fuzzy games

- 1965 (Zadeh): conception of the **Fuzzy Set Theory**
- 1974 (Aubin): cooperative games with fuzzy coalitions
- 1976 (Cook): zero-sum games with multiple goals
- 1978- (Butnariu): fuzzy 2-person non cooperative games; solution concepts for n–persons fuzzy games
- 1980 (Dubois & Prade): LR-representation for computations
- 1983 (Chanas): parametric programming in FLP
- 1984 (Buckley): uncertainty of strategies and multiple fuzzy goals
- 1986- (Ponsard): imprecise preferences and uncertain payoffs
- 1989 (Campos): solving matrix games with fuzzy payoffs based on ranking functions and LP methods.
- 1992- (Sakawa & Nishizaki): multiobjective matrix and bimatrix games with fuzzy payoffs and fuzzy goals
- 2000 (Maeda): concepts of equilibrium
- 2001 (Sakawa): large scale interactive fuzzy multiobjective programming
- 2004 (Bector et al.): LP methods for solving matrix and bimatrix games

C. Application to economics

- Production model (Owen, 1975;Molina & Tejada, 2006;Nishizaki & Sakawa, 2000): multiple DMs pool resources to produce goods in a fuzzy environment; the total revenue from selling is maximized subject to constraints -> cooperative game theory
- Management of technology (MOT) (Chen & Larbani, 2006): optimal strategies in product development of nano-materials with MADM (Multiple attribute decision making)-> parametric bimatrix game
- Production inventory model (Park, 1987; Lee & Yao, 1998; Chang, 1999; Lin & Yao, 2000; Chen & Wang & Chang, 2006): imperfect production processes ; fuzzy inventory cost, fuzzy demand and production quantity, fuzzy quality of goods
- Oligopolistic competition (Greenhut & Greenhut & Mansur, 1995): fuzzy industry size, fuzzy interdependance,

I - Fuzzy environment for games

- 1. LR-representation of fuzzy numbers (FNs)
- 2. Membership functions (MFs) shapes
- 3. Arithmetic operations on FNs
- 4. Lattice on FNs

I.1 LR-representation of FNs

Besides of the usual "true" and "false" statements, there are also vague (or fuzzy) statements in the real world of the decision making. The linguistic statements may be : "possible", "almost sure", "hardly fulfilled","approximately equal to", "considerable larger to", etc.

1. Fuzzy numbers

DEFINITION 0.1 *(Membership function of a Fuzzy number). Any vague statement $\tilde{S}$ is considered as a fuzzy subset of a universe space X with the membership function $\mu_{\tilde{S}} : X \mapsto [0,1]$ such as for any $x \in X$: $\mu_{\tilde{S}}(x) = 1$ means $\tilde{S}$ is "True" for x; $\mu_{\tilde{S}}(x) = 0$ means $\tilde{S}$ is "False" for x;$0 < \mu_{\tilde{S}}(x) < 1$ means $\tilde{S}$ is "possible" for x with the degree of possibility $\mu_{\tilde{S}}(x)$. This function is called the membership function (MF) of the fuzzy number (FN).*

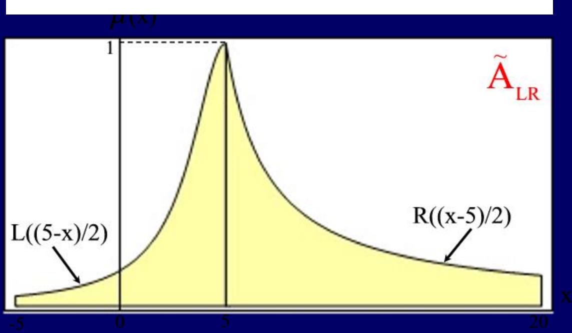
Let the MF of a FN $\tilde{A}$ be piecewise continuous triangular shaped. The FN $\tilde{A}$ is said convex normalized. The support of $\tilde{A}$ is such that $supp\,\tilde{A} = \{x \in X | \mu_{\tilde{A}}(x) = 0\}$. The height of $\tilde{A}$ is such that $hgt\,\tilde{A} = \sup_x \mu_{\tilde{A}}(x)$. The crossover points are defined by $c = \{x | \mu_{\tilde{A}}(x) = \frac{1}{2}$. The α-cuts of $\tilde{A}$ gives the crisp set of elements with at least the degree α, such that $A_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}$.

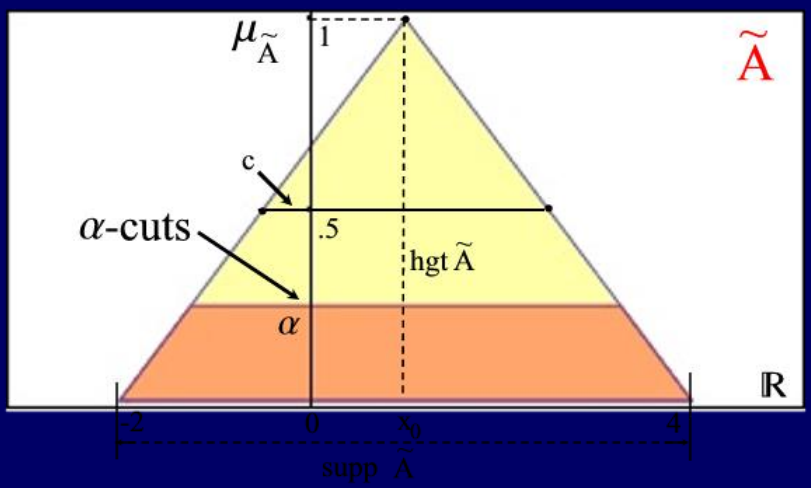
2. LR-representation

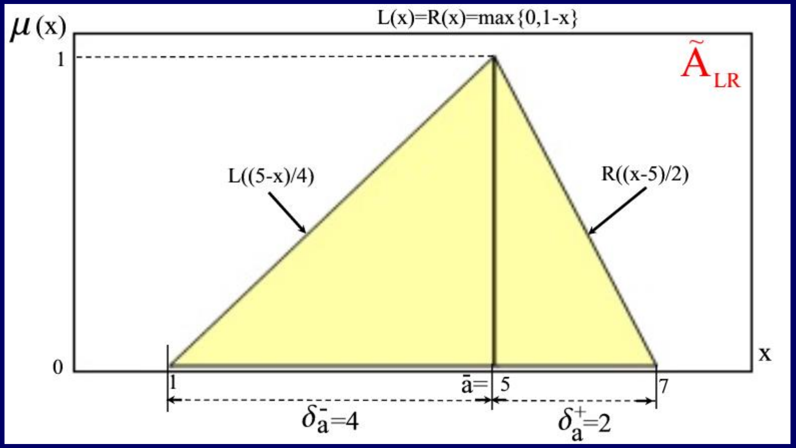
DEFINITION 0.2 *(LR-type fuzzy number). A FN $\tilde{A}_{LR} = \left(\bar{a}, \delta_a^-,\ \delta_a^+\right)$ is LR type if there exist reference functions L (for left) and R (for right, and positive scalars δ_a^-, δ_a^+ such that*

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{\bar{a}-x}{\delta_a^-}), & x \leq \bar{a} \\ R(\frac{x-\bar{a}}{\delta_a^+}), & x \geq \bar{a}, \end{cases}$$

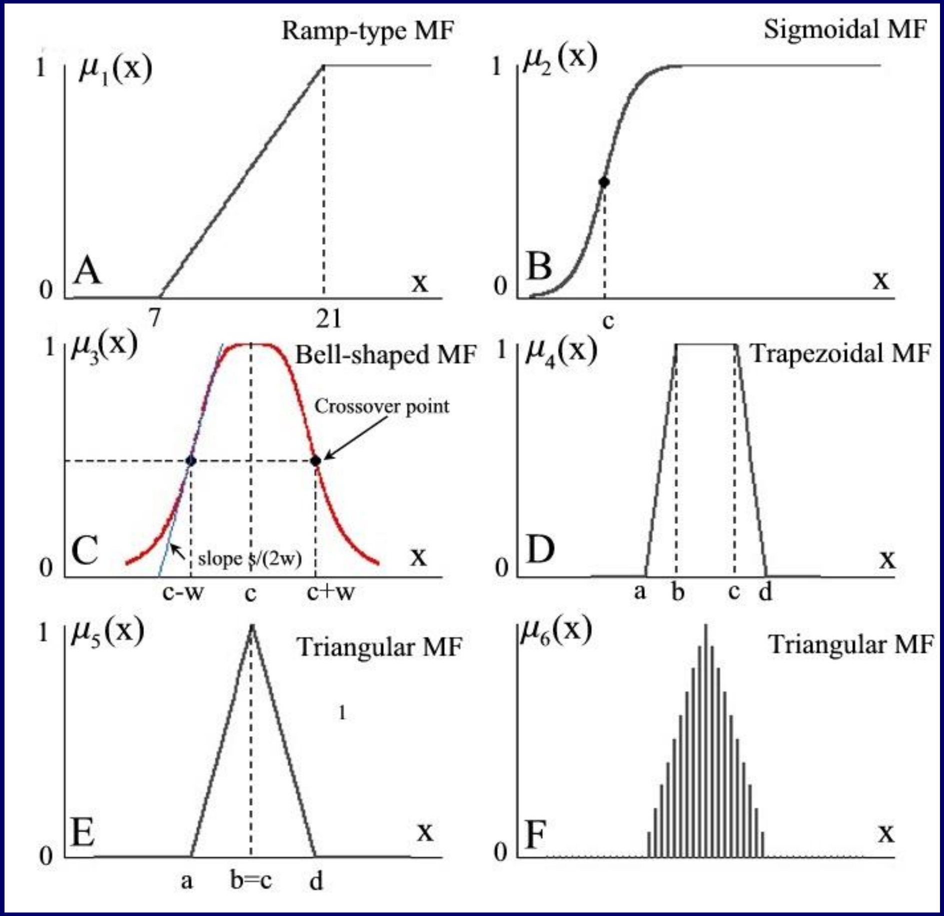
where $\bar{a}$ is the "mean value" and δ_a^-, δ_a^+ the left and right spreads. For the following example, we have the reference functions $L(x) = \frac{1}{1+x^2}$ and $R(x) = \frac{1}{1+2|x|}$ The LR-representation (Dubois & Prade, 1979) increases notably the computational efficiency.



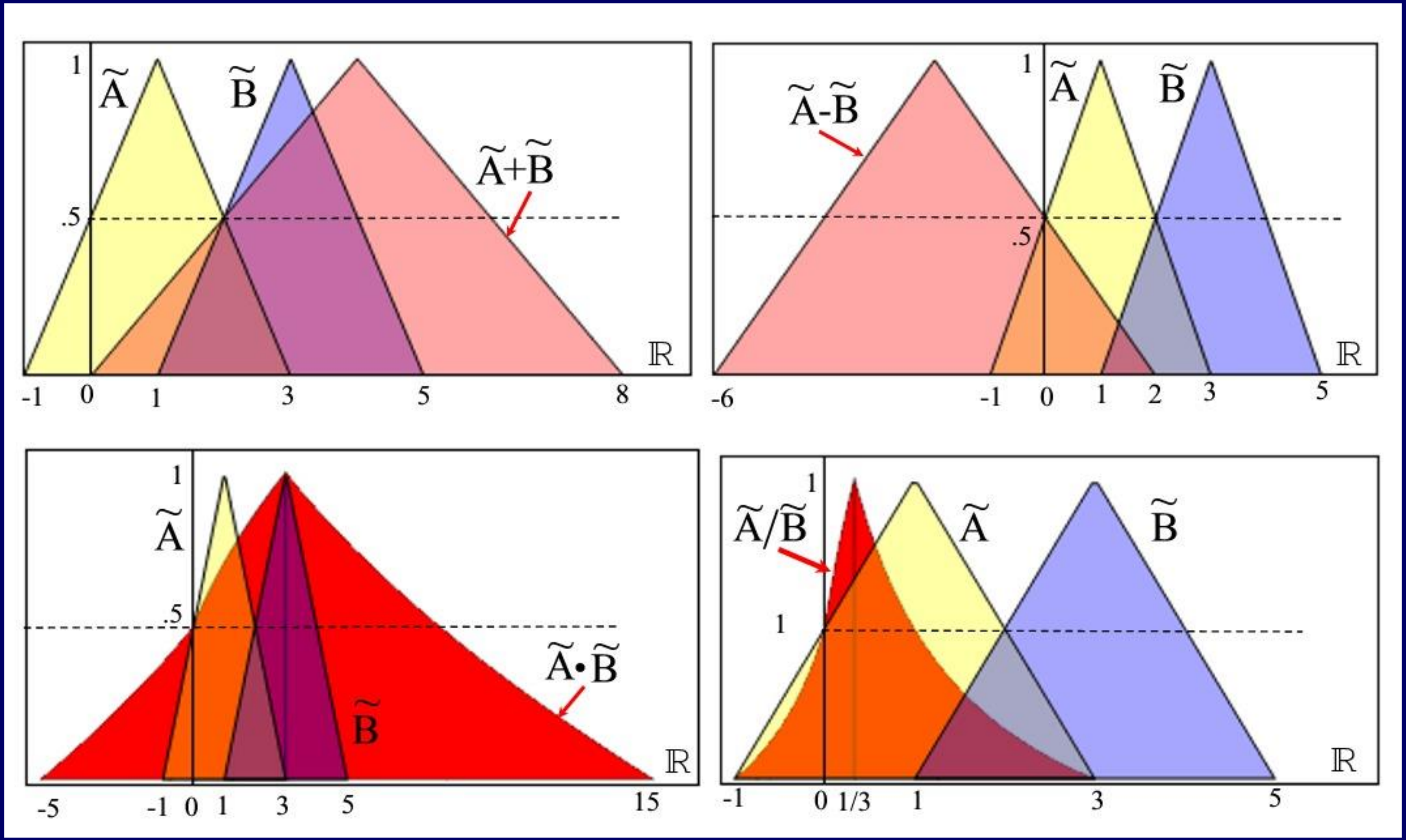




I.2 Membership function shapes



I.3 Arithmetic operations on FNs (1/2)



I.3 Arithmetic operations on FNs (2/2) : extented addition

The extension principle will give a method of calculating the MF of the output from the MFs of the input fuzzy quantities. More precisely, let $* : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ with $* \in \{+, -, ., /\}$, a binary operation over real numbers. Then, it can be extended to the operation $\circledast$ over the set $\mathfrak{F}(\mathbb{R})$ of fuzzy quantities.

1. Extension Principle

THEOREM 0.1 *(Extension principle). Denote for $a,\ b \in \mathfrak{F}(\mathbb{R})$ the quantity $c = a * b$ then the MF μ_c is derived from the continuous MFs μ_a and μ_b by the expression*

$$\mu_{a \circledast b}(z) = \sup_{z = x*y} \min\{\mu_a(x), \mu_b(y)\}$$

.

It tells that the possibility that the fuzzy quantity $c = a \circledast b$ achieves $z \in \mathbb{R}$ is as great as the most possible of the real $x,\ y$ such that $z = x * y$, where the $a,\ b$ take the values x, y respectively. For the addition, we also have the ordinary convolution

$$\mu_{a \bigoplus b}(z) = \int_0^z \mu_a(x)\mu_b(z - x)dx$$

2. Example

THEOREM 0.2 *(First Decomposition Theorem). For every $\tilde{A} \in \mathfrak{F}(\mathbb{R})$, we have $\tilde{A} = \bigcup_\alpha \tilde{A}_\alpha = \sup_\alpha$.*

Let the MFs of $\tilde{a}$ and $\tilde{b}$ be

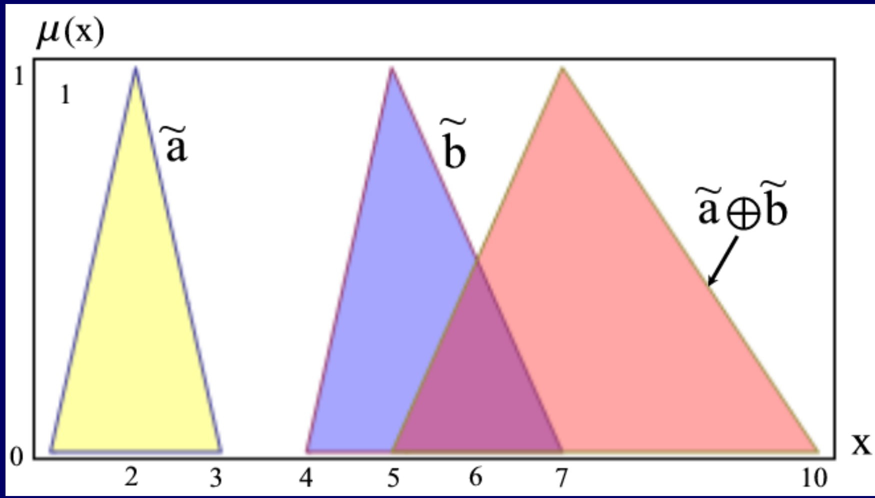
$$\mu_{\tilde{a}}(x) = \begin{cases} x - 1, & x \in [1, 2], \\ 3 - x, & x \in [2, 3], \\ 0, & \texttt{otherwise}. \end{cases}$$

and

$$\mu_{\tilde{b}}(x) = \begin{cases} x - 4, & x \in [5, 7], \\ (7 - x)/2, & x \in [7, 10], \\ 0, & \texttt{otherwise}. \end{cases}$$

The α-cuts are $\tilde{a}_\alpha = [\alpha + 1, 3 - \alpha]$ and $\tilde{b}_\alpha = [\alpha + 4, 7 - 2\alpha]$. Then, we have $\tilde{c}_\alpha = (\tilde{a} \bigoplus \tilde{b})_\alpha = \tilde{a}_\alpha + \tilde{b}_\alpha = [2\alpha + 5, 10 - 3\alpha]$. Solving in α, we obtain

$$\mu_{\tilde{a} \bigoplus \tilde{b}}(x) = \begin{cases} (x - 5)/2, & x \in [5, 7], \\ (10 - x)/3, & x \in [7, 10], \\ 0, & \texttt{otherwise}. \end{cases}$$

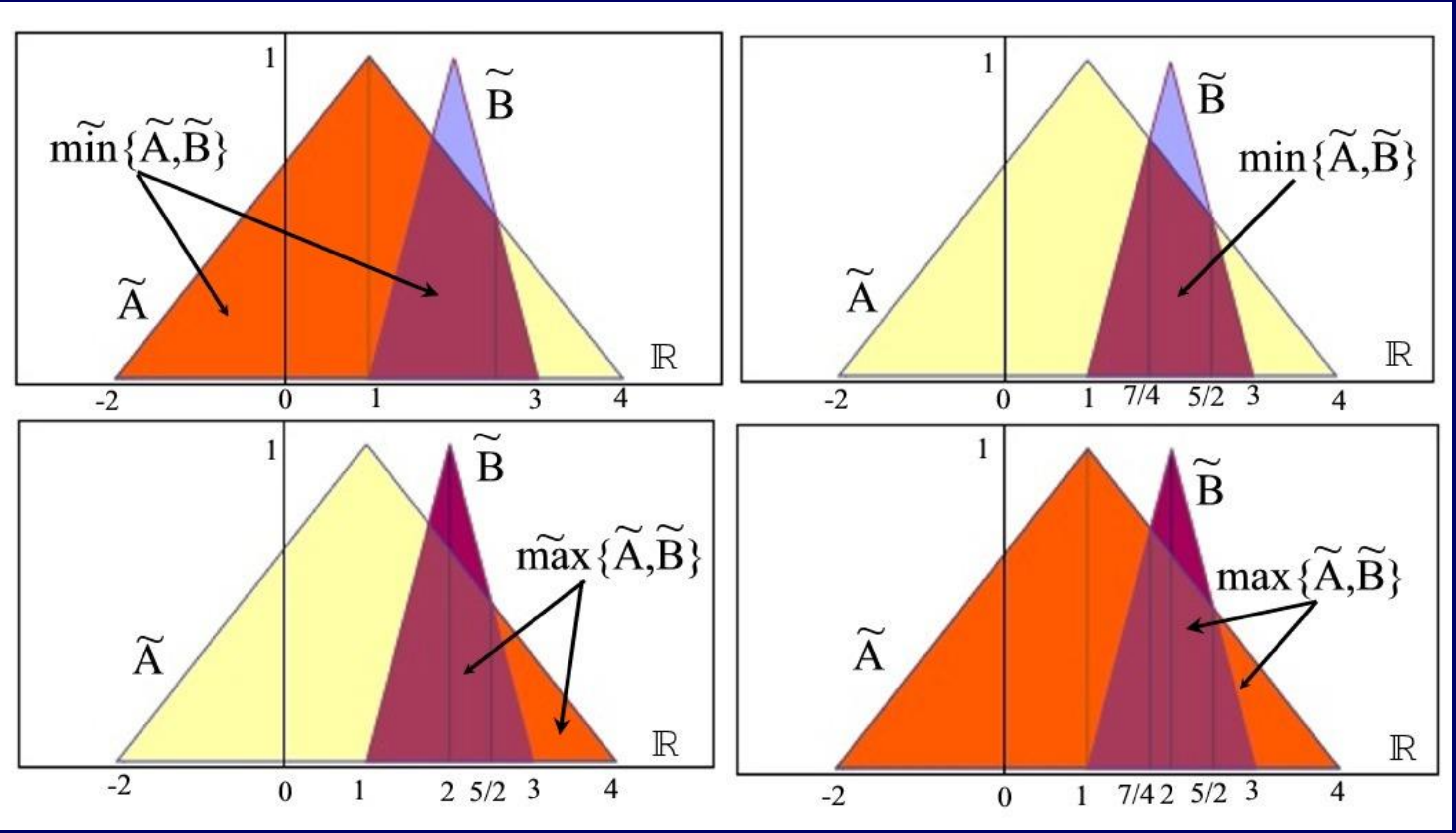


3. Note

THEOREM 0.3 *(Addition of LR-type FNs). Let $\tilde{a}$ and $\tilde{b}$ two FNs of LR-type $\tilde{a} = (\bar{a}, \delta_a^-, \delta_a^+)_{LR}$ and $\tilde{b} = (\bar{b}, \delta_b^-, \delta_b^+)_{LR}$, then $\tilde{a} \bigoplus \tilde{b} = (\bar{a} + \bar{b}, \delta_a^- + \delta_b^-, \delta_a^+ + \delta_b^+)_{LR}$.*

For $\tilde{a} = (2, 1, 1)_{LR}$ and $\tilde{b} = (5, 1, 2)_{LR}$, we simply have $\tilde{a} \bigoplus \tilde{b} = (7, 2, 3)_{LR}$.

I.4 Lattice on FNs



II – Single objective fuzzy two person matrix games

- 1. Equivalence theorems
- 2. Fuzzy LP problems
- 3. Matrix games with fuzzy payoffs
- 4. Matrix games with a fuzzy goal

I.1 Equivalence theorems

1. Equivalence theorems

Bimatrix game

Two players I and II have mixed strategies given by the n-dimensional vector $\mathbf{x}$ and the m-dimensional vector $\mathbf{y}$, respectively. The payoffs of players I and II are the $n \times m$ matrices $\mathbf{A}$ and $\mathbf{B}$, respectively. Let $\mathbf{e}_n$ be an n-dimensional vector of ones, $\mathbf{e}_m$ having a dimension m. The objective of player I will be: $\{\max_{\mathbf{x}} \mathbf{x}'\mathbf{A}\mathbf{y}$ subject to $\mathbf{e}'_n\mathbf{x} = 1,\ x \geq 0\}$. The objective of player II will then be: $\{\max_{\mathbf{y}} \mathbf{x}'\mathbf{B}\mathbf{y}$ subject to $\mathbf{e}'_m\mathbf{y} = 1,\ \mathbf{y} \geq 0\}$.

Equivalence to QP problems

DEFINITION 0.1 *Nash equilibrium. A Nash equilibrium point is is a pair of strategies $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ such that the objectives of the two players are full filled simultaneously. We have*

$$\begin{aligned} \boldsymbol{x}'^*\boldsymbol{A}\boldsymbol{y}^* &= \max_{\boldsymbol{x}}\{\boldsymbol{x}'\boldsymbol{A}\boldsymbol{y}^* | e_n\boldsymbol{x} = 1, \boldsymbol{x} \geq 0\} \\ \boldsymbol{x}'^*\boldsymbol{A}\boldsymbol{y}^* &= \max_{\boldsymbol{y}}\{\boldsymbol{x}'^*\boldsymbol{A}\boldsymbol{y} | e_m\boldsymbol{y} = 1, \boldsymbol{y} \geq 0\} \end{aligned}$$

Applying the Kuhn-Tucker necessary and sufficient conditions, we have

THEOREM 0.2 *(Equivalence Theorem). A necessary and sufficient condition that $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ be an equilibrium point is it is the solution of the QP problem*

$$\max_{\boldsymbol{x},\boldsymbol{y},\alpha,\beta} \quad \boldsymbol{x}'(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{y} - \alpha - \beta$$

$$\begin{aligned} &\texttt{subject to} \\ \boldsymbol{A}\boldsymbol{y} &\leq \alpha \boldsymbol{e}_n, \\ \boldsymbol{B}'\boldsymbol{x} &\leq \beta \boldsymbol{e}_m, \\ \boldsymbol{e}'_n\boldsymbol{x} &= 1, \\ \boldsymbol{e}'_m\boldsymbol{y} &= 1, \\ \boldsymbol{x} \geq 0,\ \boldsymbol{y} &\geq 0, \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ are the negative of the multipliers associated with the constraints.

PROOF. see O.L. Mangasarian and H. Stone (1964),pp. 350-351.□

Equivalence to LP problems

In zero-sum games we have $\mathbf{B} = -\mathbf{A}$. The QP problems degenerate to two dual problems. We have

$$\begin{aligned} &\max_{\mathbf{x},\gamma} \quad \gamma \\ &\text{subject to} \\ -\mathbf{A}'\mathbf{x} &\leq -\gamma\mathbf{e}_n, \\ \mathbf{e}'_n\mathbf{x} &= 1, \\ \mathbf{x} &\geq 0. \end{aligned}$$

and

$$\begin{aligned} &\min_{\mathbf{x},\alpha} \quad \alpha \\ &\text{subject to} \\ \mathbf{A}\mathbf{y} &\leq \alpha\mathbf{e}_m, \\ \mathbf{e}'_m\mathbf{y} &= 1, \\ \mathbf{y} &\geq 0. \end{aligned}$$

Numerical example with *Mathematica*:

```
Clear[A, B];

A := {{2, -1}, {-1, 1}}; B := {{1, -1}, {-1, 2}}; x := {x1, x2}; y := {y1, y2}; e := {1, 1}

Timing[Maximize[Rationalize[
  {x.(A+B).y - α - β,
   A[[1]].y ≤ α && A[[2]].y ≤ α &&
    Transpose[B][[1]].x ≤ β && Transpose[B][[2]].x ≤ β &&
    e.x == 1 && e.y == 1 &&
    x1 ≥ 0 && x2 ≥ 0 && y1 ≥ 0 && y2 ≥ 0}],
  {x1, x2, y1, y2, α, β}]]
```

$$\left\{61.375, \left\{0, \left\{x1 \to \frac{3}{5}, x2 \to \frac{2}{5}, y1 \to \frac{2}{5}, y2 \to \frac{3}{5}, \alpha \to \frac{1}{5}, \beta \to \frac{1}{5}\right\}\right\}\right\}$$

```
N[%]

{61.375, {0., {x1 → 0.6, x2 → 0.4, y1 → 0.4, y2 → 0.6, α → 0.2, β → 0.2}}}
```

II.2 Standard fuzzy LP problem

1. LP problem with fuzzy constraint

The fuzzy linear programming problem consists of a crisp objective function and a fuzzy constraint, such that

$$\begin{aligned} \max_{\mathbf{x}} \quad & \mathbf{c}^{\top}.\mathbf{x} \quad (\mathbf{c}, \mathbf{x} \in \mathbb{R}^n) \\ & \text{subject to} \\ \tilde{\mathbf{A}}_i.\mathbf{x} \ & \precsim \tilde{b}_i, \ (i \in \mathbb{N}_m) \\ & \mathbf{x} \geq 0. \end{aligned}$$

The $\tilde{a}_{ij}$, $\tilde{b}_i$ are fuzzy numbers of $\mathfrak{F}(\mathbb{R})$ whose values are known with imprecision. The fuzzy inequality $\precsim$ tells that the decision maker (DM) will allow some violation in the accomplishment of the constraint. The membership functions $\mu_i : \mathfrak{F}(\mathbb{R}) \mapsto [0,1]$, $i \in \mathbb{N}_m$ will measure the adequacy between both sides of the constraint $\tilde{A}_i.x$ and $\tilde{b}_i$. The FNs $\tilde{p}_i$ will express the margins of tolerance for each constraint.

2. Converting the fuzzy constraint

The FLP problem may be written

$$\begin{aligned} \max_{\mathbf{x}} \quad & \mathbf{c}^{\top}.\mathbf{x} \quad (\mathbf{c}, \mathbf{x} \in \mathbb{R}^n) \\ & \text{subject to} \\ \tilde{\mathbf{A}}_i.\mathbf{x} \ & \leq_{\Re} \tilde{b}_i + \tilde{p}_i(1-\alpha), \ (i \in \mathbb{N}_m) \\ & \mathbf{x} \geq 0. \end{aligned}$$

In the constraint, the inequality rule $\leq_{\Re}$ is to be chosen by the DM among several ranking functions (or index) matching each FN into the real line, such that $F : \mathfrak{F}(\mathbb{R}) \mapsto \mathbb{R}$.

3. Solving one auxiliary problem

Let a triangular FN be expressed by $\tilde{a} = (a, a^-, a^+)$, where a^-, a^+ are the lower and the upper limit of the support, respectively. Ranking the two fuzzy sides of the inequality may give the following auxiliary parametric LP problem

$$\begin{aligned} \max_{\mathbf{x}} \quad & \mathbf{c}^{\top}.\mathbf{x} \quad (\mathbf{c}, \mathbf{x} \in \mathbb{R}^n) \\ & \text{subject to} \\ (A_i + A_i^- + A_i^+).\mathbf{x} \ & \leq (b_i + b_i^- + b_i^+) + (p_i + p_i^- + p_i^+)(1-\alpha), \ (i \in \mathbb{N}_m) \\ & \mathbf{x} \geq 0. \end{aligned}$$

4. Numerical example

This numerical example is due to Delgado et al (1990). The FLP problem is

$$\begin{aligned} \max_{x_1,x_2} \quad & z = 5x_1 + 6x_2 \\ & \text{subject to} \\ \tilde{3}x_1 + \tilde{4}x_2 & \precsim \tilde{18}, \\ \tilde{2}x_1 + \tilde{1}x_2 & \precsim \tilde{7}, \\ & x_1, x_2 \geq 0. \end{aligned}$$

According to the rule that the DM will choose, two different auxiliary problems and solutions are obtained. We have

rule 1 : $\bar{x} <_{\Re_1} \bar{y} \Leftrightarrow x \leq y$.

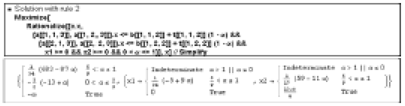
$$\begin{aligned} \max_{x_1,x_2} \quad & z = 5x_1 + 6x_2 \\ & \text{subject to} \\ 3x_1 + 4x_2 & \leq 18 + 3(1-\alpha), \\ 2x_1 + x_2 & \leq 7 + (1-\alpha), \\ & x_1, x_2 \geq 0, \ \alpha \in (0,1] \end{aligned}$$

The parametrized solution with rule 1 given by Mathematica is shown hereafter.

rule 2 : $\bar{x} <_{\Re_2} \bar{y} \Leftrightarrow \bar{x} \leq \underline{y}$.

$$\begin{aligned} \max_{x_1,x_2} \quad & z = 5x_1 + 6x_2 \\ & \text{subject to} \\ 4x_1 + 5.5x_2 & \leq 16 + 2.5(1-\alpha), \\ 3x_1 + 2x_2 & \leq 6 + .5(1-\alpha), \\ & x_1, x_2 \geq 0, \ \alpha \in (0,1] \end{aligned}$$

The parametrized solution with rule 2 given by Mathematica is shown hereafter.



II.3 Matrix Game with fuzzy payoffs (1/2)

1. Problem of the Player I

The fuzzy matrix game problem of the Player I in a fuzzy environment is

$$\max\ v$$
subject to
$$\sum_{i=1}^{m} \bar{a}_{ij} x_i \gtrsim v,\ (j \in \mathbb{N}_n)$$
$$\sum_{i=1}^{m} x_i = 1,\ x_i \geq 0\ (i \in \mathbb{N}_m).$$

The payoffs of Player I $\bar{a}_{ij}$ are fuzzy numbers of $\mathfrak{F}(\mathbb{R})$ whose values are known with imprecision. The fuzzy inequality $\precsim$ tells that the decision maker (DM) will allow some violation in the accomplishment of the constraint.

2. Classical changing of the variables

Let change the variables into $u_i = \frac{x_i}{v}\ (i \in \mathbb{N}_m)$. We have $\sum_{i=1}^{m} u_i = \frac{\sum_{i=1}^{m} x_i}{v} = \frac{1}{v}$, then $v = \frac{1}{\sum_{i=1}^{m} u_i}$. The initial problem is transformed to

$$\min \sum_{i=1}^{m} u_i$$
subject to
$$\sum_{i=1}^{m} \bar{a}_{ij} u_i \gtrsim 1,\ (j \in \mathbb{N}_n)$$
$$u_i \geq 0\ (i \in \mathbb{N}_m).$$

3. Introducing ranking functions

For solving the FLP problem in canonical form, we apply the following procedure : ranking functions are introduced to compare both fuzzy sides of the inequality, and solving a parametric LP problem. The problem of the player I is transformed to

$$\min \sum_{i=1}^{m} u_i$$
subject to
$$\sum_{i=1}^{m} \bar{a}_{ij} u_i \geq_{\mathfrak{R}} 1 - \bar{p}_j(1-\alpha),\ (j \in \mathbb{N}_n)$$
$$u_i \geq 0\ (i \in \mathbb{N}_m),\ \alpha \in (0,1].$$

The FNs $\bar{p}_i$'s are the maximum violation that the player will allow for the constraints.

6. Problem of the Player II

The fuzzy matrix game problem of the Player II in a fuzzy environment is

$$\min\ w$$
subject to
$$\sum_{j=1}^{n} \bar{a}_{ij} y_j \gtrsim w,\ (i \in \mathbb{N}_m)$$
$$\sum_{j=1}^{n} y_j = 1,\ y_j \geq 0\ (j \in \mathbb{N}_n).$$

The losses of Player II $\bar{a}_{ij}$ are fuzzy numbers of $\mathfrak{F}(\mathbb{R})$ whose values are known with imprecision. The fuzzy inequality $\lesssim$ tells that the decision maker (DM) will allow some violation in the accomplishment of the constraint.

7. Classical changing of the variables

Let change the variables into $s_j = \frac{y_j}{w}\ (j \in \mathbb{N}_n)$. We have $\sum_{j=1}^{n} s_j = \frac{\sum_{j=1}^{n} y_j}{w} = \frac{1}{w}$, the $w = \frac{1}{\sum_{j=1}^{n} s_j}$. The initial problem is transformed to

$$\max \sum_{j=1}^{n} s_i$$
subject to
$$\sum_{j=1}^{n} \bar{a}_{ij} s_j \lesssim 1,\ (i \in \mathbb{N}_m)$$
$$s_j \geq 0\ (j \in \mathbb{N}_n).$$

The RHS of the fuzzy inequality is transformed to a crisp number.

8. Introducing ranking functions

For solving the FLP problem in canonical form, we apply the following procedure : ranking functions are introduced to compare both fuzzy sides of the inequality, and solving a parametric LP problem. The problem of the player II is transformed to

$$\max \sum_{j=1}^{m} s_i$$
subject to
$$\sum_{j=1}^{n} \bar{a}_{ij} s_j \leq_{\mathfrak{R}} 1 + \bar{q}_i(1-\alpha),\ (i \in \mathbb{N}_m)$$
$$s_j \geq 0\ (j \in \mathbb{N}_n),\ \alpha \in (0,1].$$

The FNs $\bar{q}_j$'s are the maximum violation that the player will allow for the constraints.

II.3 Matrix game with fuzzy payoffs (2/2)

4. Solving the auxiliary problem

Let the payoffs $\tilde{a}_{ij}$ of Player I be a triangular FNs be expressed by $\tilde{a}_{ij} = (a_{ij}, a_{ij}^-, a_{ij}^+)$, where a_{ij}^-, a_{ij}^+ are the lower and the upper limit of the support, respectively. Ranking the two sides of the inequality may render the following auxiliary parametric LP problem

$$\min \sum_{i=1}^{m} u_i$$

subject to

$$\sum_{i=1}^{m} (a_{ij} + a_{ij}^- + a_{ij}^+) u_i \geq 3 + (p_i + p_i^- + p_i^+)(1-\alpha), \ (j \in \mathbb{N}_n)$$

$$u_i \geq 0 \ (i \in \mathbb{N}_m), \ \alpha \in (0,1].$$

5. Numerical example

This numerical example is due to Campos (1989). The fuzzy payoff matrix of Player I is

$$\tilde{A} = \begin{pmatrix} \widetilde{180} & \widetilde{156} \\ \widetilde{90} & \widetilde{180} \end{pmatrix}$$

The FNs are defined by $\widetilde{180} = (180, 175, 190)$, $\widetilde{156} = (156, 150, 158)$, $\widetilde{90} = (90, 80, 100)$. The fuzzy margins are $\tilde{p}_1 = \tilde{p}_2 = (0.10, 0.08, 0.11)$ for the Player I.

FLP problem

$$\min \ u_1 + u_2$$

subject to

$$\widetilde{180} u_1 + \widetilde{90} u_2 \geq_{\mathcal{R}} 1 - \widetilde{0.10}(1-\alpha)$$

$$\widetilde{90} u_1 + \widetilde{180} u_2 \geq_{\mathcal{R}} 1 - \widetilde{0.10}(1-\alpha)$$

$$u_1, \ u_2 \geq 0, \ \alpha \in (0,1].$$

Auxiliary problem

$$\min \ u_1 + u_2$$

subject to

$$545 u_1 + 270 u_2 \geq 3 - 0.29(1-\alpha)$$

$$464 u_1 + 545 u_2 \geq 3 - 0.29(1-\alpha)$$

$$u_1, \ u_2 \geq 0, \ \alpha \in (0,1].$$

Solution

We have $x^* = (0.77, 0.23)$ and $v(\alpha) = \frac{482.43}{3 - 0.29(1-\alpha)}$, $\alpha \in (0,1]$.

9. Solving the auxiliary problem

Let the losses $\tilde{a}_{ij}$ of Player II be a triangular FNs be expressed by $\tilde{a}_{ij} = (a_{ij}, a_{ij}^-, a_{ij}^+)$, where a_{ij}^-, a_{ij}^+ are the lower and the upper limit of the support, respectively. Ranking the two fuzzy sides of the inequality may render the following auxiliary parametric LP problem

$$\max \sum_{j=1}^{n} s_i$$

subject to

$$\sum_{j=1}^{n} (a_{ij} + a_{ij}^- + a_{ij}^+) s_j \leq 3 + (q_i + q_i^- + q_i^+)(1-\alpha), \ (i \in \mathbb{N}_m)$$

$$s_j \geq 0 \ (j \in \mathbb{N}_n), \ \alpha \in (0,1].$$

10. Numerical example

The fuzzy losses matrix of Player II is

$$\tilde{A} = \begin{pmatrix} \widetilde{180} & \widetilde{156} \\ \widetilde{90} & \widetilde{180} \end{pmatrix}$$

The FNs are defined by $\widetilde{180} = (180, 175, 190)$, $\widetilde{156} = (156, 150, 158)$, $\widetilde{90} = (90, 80, 100)$. The fuzzy margins are $\tilde{q}_1 = \tilde{q}_2 = (0.15, 0.14, 0.17)$ for the Player II.

FLP problem

$$\max \ s_1 + s_2$$

subject to

$$\widetilde{180} s_1 + \widetilde{156} s_2 \leq_{\mathcal{R}} 1 - \widetilde{0.15}(1-\alpha)$$

$$\widetilde{90} s_1 + \widetilde{180} s_2 \leq_{\mathcal{R}} 1 - \widetilde{0.15}(1-\alpha)$$

$$s_1, \ s_2 \geq 0, \ \alpha \in (0,1].$$

Auxiliary problem

$$\max \ s_1 + s_2$$

subject to

$$545 s_1 + 464 s_2 \leq 3 + 0.46(1-\alpha)$$

$$270 s_1 + 545 s_2 \leq 3 + 0.46(1-\alpha)$$

$$s_1, \ s_2 \geq 0, \ \alpha \in (0,1].$$

Solution

We have $y^* = (0.23, 0.77)$ and $w(\alpha) = \frac{482.43}{3 + 0.49(1-\alpha)}$, $\alpha \in (0,1]$.

II.4 Matrix games with a fuzzy goal (1/2)

Let the single-objective matrix game $G = (S^m, S^n, \mathbf{A})$ with fuzzy goals where S^m and S^n denote the compact convex strategy space of Players, such that $S^m = \{\mathbf{x} \in \mathbb{R}^m_+, \mathbf{e}'\mathbf{x} = 1\}$ and $S^n = \{\mathbf{y} \in \mathbb{R}^n_+, \mathbf{e}'\mathbf{y} = 1\}$, and where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the payoff matrix of the game with real entries.

1. Fuzzy goal of the Player I

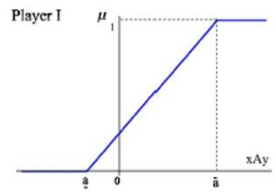
For any pair od strategies (x, y) a membership function $\mu(x, y)$ depends on the expected payoff **xAy**. Assume that the degree of satisfaction increases linearly, we have

$$\mu(\mathbf{xAy}) = \begin{cases} 1, & \mathbf{xAy} \geq \bar{a} \\ \frac{\mathbf{xAy} - \underline{a}}{\bar{a} - \underline{a}}, & \underline{a} \leq \mathbf{xAy} \leq \bar{a} \\ 0, & \mathbf{xAy} \leq \underline{a}. \end{cases}$$

where $\bar{a}$ and $\underline{a}$ are the best and the worst degree of satisfaction to the Player I, respectively. These extremal values are determined by

$$\begin{aligned} \bar{a} &= \max_x \max_y \mathbf{xAy} = \max_{i \in \mathbb{N}_m} \max_{j \in \mathbb{N}_n} a_{ij} \\ \underline{a} &= \min_x \min_y \mathbf{xAy} = \min_{i \in \mathbb{N}_m} \min_{j \in \mathbb{N}_n} a_{ij} \end{aligned}$$

The membership function of the fuzzy goal is of the shape



2. Player I's maximin solution

THEOREM 0.1 *(Maximin solution). For a single-objective two person matrix game with a linearly fuzzy goal function, the Player I's maximin solution w.r.t. a degree of achievement of the fuzzy goal is equal to an optimal solution to the LP problem*

$$\begin{aligned} &\max \lambda \\ &\textit{subject to} \\ &\frac{1}{\bar{a} - \underline{a}}\left(\sum_{i=1}^{m} a_{ij} x_i - \underline{a}\right) \geq \lambda, (j \in \mathbb{N}_n) \\ &\boldsymbol{e}'\boldsymbol{x} = 1, \ (\boldsymbol{e}, \boldsymbol{x} \in \mathbb{R}^m), \\ &\boldsymbol{x} \geq 0. \end{aligned}$$

Proof : Nishizaki and Sakawa (2001), p. 39. □

4. Fuzzy goal of the Player II

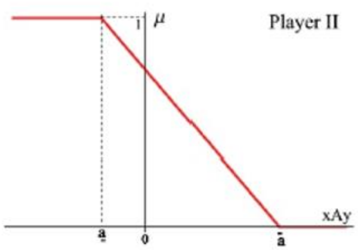
For any pair od strategies (x, y) a membership function $\mu(x, y)$ depends on the expected payoff **xAy**. Assume that the degree of satisfaction increases linearly, we have

$$\mu(\mathbf{xAy}) = \begin{cases} 1, & \mathbf{xAy} \leq \underline{a} \\ \frac{\bar{a} - \mathbf{xAy}}{\bar{a} - \underline{a}}, & \underline{a} \leq \mathbf{xAy} \leq \bar{a} \\ 0, & \mathbf{xAy} \geq \bar{a}, \end{cases}$$

where $\bar{a}$ and $\underline{a}$ are the worst and the best degree of satisfaction to the Player II, respectively. These extremal values are determined by

$$\begin{aligned} \bar{a} &= \max_x \max_y \mathbf{xAy} = \max_{i \in \mathbb{N}_m} \max_{j \in \mathbb{N}_n} a_{ij} \\ \underline{a} &= \min_x \min_y \mathbf{xAy} = \min_{i \in \mathbb{N}_m} \min_{j \in \mathbb{N}_n} a_{ij} \end{aligned}$$

The membership function of the fuzzy goal is of the shape



Proof : Nishizaki and Sakawa (2001), p. 41. □

5. Player II's minimax solution

THEOREM 0.2 *(Minimax solution). For a single-objective two person matrix game with a linearly fuzzy goal function, the Player II's minimax solution w.r.t. a degree of achievement of the fuzzy goal is equal to an optimal solution to the LP problem*

$$\begin{aligned} &\min \lambda \\ &\textit{subject to} \\ &\frac{1}{\bar{a} - \underline{a}}\left(\sum_{j=1}^{n} a_{ij} y_j - \underline{a}\right) \leq \lambda + 1, (i \in \mathbb{N}_m) \\ &\boldsymbol{e}'\boldsymbol{y} = 1, \ (\boldsymbol{e}, \boldsymbol{y} \in \mathbb{R}^n), \\ &\boldsymbol{y} \geq 0. \end{aligned}$$

II.4 Matrix game with a fuzzy goal (2/2)

3. Numerical example

In the Cook's example, a 3×3 payoff matrix is given by

$$A = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{pmatrix}$$

We have $\bar{a} = 6$ and $\underline{a} = -2$. Then, we have to solve the LP problem

$$\begin{aligned} &\max_{\mathbf{x},\lambda} \ \lambda \\ &\text{subject to} \\ &2x_1 - x_2 + 2 \geq 8\lambda, \\ &5x_1 - 2x_2 + 3x_3 + 2 \geq 8\lambda, \\ &x_1 + 6x_2 - x_3 + 2 \geq 8\lambda, \\ &\mathbf{e}.\mathbf{x} = 1, \ \mathbf{x} \geq 0. \end{aligned}$$

The results of Player I are

Optimality with fuzzy goal (Player I)

```
NMaximize[{λ,
  (B[[1]].x - aw) / (ab - aw) ≥ λ, (B[[2]].x - aw) / (ab - aw) ≥ λ, (B[[3]].x - aw) / (ab - aw) ≥ λ,
  x1 + x2 + x3 == 1, x1 ≥ 0, x2 ≥ 0, x3 ≥ 0}, {x1, x2, x3, λ}]
```

```
{0.453125, {x1 → 0.875, x2 → 0.125, x3 → 0., λ → 0.453125}}
```

6. Numerical example

In the Cook's example, a 3×3 payoff matrix is given by

$$A = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{pmatrix}$$

We have $\bar{a} = 6$ and $\underline{a} = -2$. Then we have to solve the LP problem

$$\begin{aligned} &\min_{\mathbf{y},\lambda} \ \lambda \\ &\text{subject to} \\ &2y_1 + 5y_2 + y_3 + 2 \leq 8(1+\lambda), \\ &-y_1 - 2y_2 + 6x_3 + 2 \leq 8(1+\lambda), \\ &3y_2 - y_3 + 2 \leq 8(1+\lambda), \\ &\mathbf{e}.\mathbf{y} = 1, \ \mathbf{y} \geq 0. \end{aligned}$$

The results of Player I are

Optimality with fuzzy goal (Player II)

```
NMinimize[{λ,
  (A[[1]].y - aw) / (ab - aw) ≤ λ + 1, (A[[2]].y - aw) / (ab - aw) ≤ λ + 1, (A[[3]].y - aw) / (ab - aw) ≤ λ + 1,
  y1 + y2 + y3 == 1, y1 ≥ 0, y2 ≥ 0, y3 ≥ 0}, {y1, y2, y3, λ}]
```

```
{-0.546875, {y1 → 0.625, y2 → 0., y3 → 0.375, λ → -0.546875}}
```

III- Multiobjective fuzzy two person matrix game

- 1. Multiobjective matrix game with fuzzy goals
- 2. Multiobjective matrix game with fuzzy payoffs and fuzzy goals

III.1 Multiobjective matrix game with fuzzy goals (1/2)

Let the multi-objective matrix game $G = (S^m, S^n, \mathbf{A}_1, \ldots, \mathbf{A}_r)$ with fuzzy goals, where S^m and S^n denote the compact convex strategy spaces of Players, such that $S^m = \{\mathbf{x} \in \mathbb{R}^m_+, \mathbf{e}'\mathbf{x} = 1\}$ and $S^n = \{\mathbf{y} \in \mathbb{R}^n_+, \mathbf{e}'\mathbf{y} = 1\}$, and where the $\mathbf{A}'_k$s $\in \mathbb{R}^{m \times n}$ are the payoff matrices of the game with real entries.

1. Fuzzy goals of the Player I

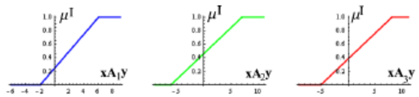
The Player is supposed to have a fuzzy goal for each of the objectives. For any pair od strategies $(\mathbf{x}, \mathbf{y})$ a membership function $\mu_k(\mathbf{x}, \mathbf{y})$ depends on the expected payoff $\mathbf{x}\mathbf{A}_k\mathbf{y}$. Assume that the degree of satisfaction increases linearly, we have

$$\mu(\mathbf{x}\mathbf{A}_k\mathbf{y}) = \begin{cases} 1, & \mathbf{x}\mathbf{A}_k\mathbf{y} \geq \bar{a}_k \\ \frac{\mathbf{x}\mathbf{A}_k\mathbf{y} - \underline{a}_k}{\bar{a}_k - \underline{a}_k}, & \underline{a}_k \leq \mathbf{x}\mathbf{A}_k\mathbf{y} \leq \bar{a}_k \\ 0, & \mathbf{x}\mathbf{A}_k\mathbf{y} \leq \underline{a}_k, \end{cases}$$

where $\bar{a}_k$ and $\underline{a}_k$ are the best and the worst degree of satisfaction to the Player I, respectively. These extremal values are determined by

$$\begin{aligned} \bar{a}_k &= \max_{\mathbf{x}} \max_{\mathbf{y}} \mathbf{x}\mathbf{A}_k\mathbf{y} = \max_{i \in \mathbb{N}_m} \max_{j \in \mathbb{N}_n} a_{k,ij} \\ \underline{a}_k &= \min_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x}\mathbf{A}_k\mathbf{y} = \min_{i \in \mathbb{N}_m} \min_{j \in \mathbb{N}_n} a_{k,ij} \end{aligned}$$

The membership function of the fuzzy goals is of the shape



2. Bellman-Zadeh fuzzy decision rule

According to the Bellman-Zadeh symmetry principle, afuzzy decision set is achieved by using an appropriate aggregation of the fuzzy sets.

Definition 0.1 *Let X be a set of possible actions, $\{\bar{G}_j(j \in \mathbb{N}_n\}$ a set of fuzzy objectives, and $\{\bar{C}_i(j \in \mathbb{N}_m\}$ the decision set is defined by*

$$\bar{D} = \left(\bigcap_{j=1}^{n} \bar{G}_j\right) \bigcap \left(\bigcap_{i=1}^{m} \bar{C}_i\right)$$

with MF $\mu_{\bar{D}} : X \mapsto [0, 1]$ given by

$$\mu_{\bar{D}}(x) = \left(\bigwedge_{j=1}^{n} \mu_{\bar{G}_j}(x)\right) \bigwedge \left(\bigwedge_{i=1}^{m} \mu_{\bar{C}_i}(x)\right)$$

.

The MFs of the aggregate fuzzy goal can be expressed as

$$\mu(x, y) = \min_{k \in \mathbb{N}_r} \left\{ \mu_k(\mathbf{x}\mathbf{A}_k\mathbf{y}) \right\}$$

. Hence, we have with linear MFs

$$\mu(x, y) = \min_{k \in \mathbb{N}_r} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{a_{k,ij}}{\bar{a}_k - \underline{a}_k} x_i y_j - \frac{\underline{a}_k}{\bar{a}_k - \underline{a}_k} \right\}$$

.

5. Fuzzy goals of the Player II

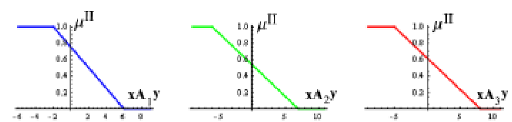
The Player is supposed to have a fuzzy goal for each of the objective. For any pair od strategies $(\mathbf{x}, \mathbf{y})$ a membership function $\mu_k(\mathbf{x}, \mathbf{y})$ depends on the expected payoff $\mathbf{x}\mathbf{A}_k\mathbf{y}$. Assume that the degree of satisfaction decreases linearly, we have

$$\mu(\mathbf{x}\mathbf{A}_k\mathbf{y}) = \begin{cases} 1, & \mathbf{x}\mathbf{A}_k\mathbf{y} \leq \underline{a}_k \\ \frac{\bar{a}_k - \mathbf{x}\mathbf{A}_k\mathbf{y}}{\bar{a}_k - \underline{a}_k}, & \underline{a}_k \leq \mathbf{x}\mathbf{A}_k\mathbf{y} \leq \bar{a}_k \\ 0, & \mathbf{x}\mathbf{A}_k\mathbf{y} \geq \bar{a}_k, \end{cases}$$

where $\bar{a}_k$ and $\underline{a}_k$ are the worst and the best degree of satisfaction to the Player II, respectively. These extremal values are determined by

$$\begin{aligned} \bar{a}_k &= \max_{\mathbf{x}} \max_{\mathbf{y}} \mathbf{x}\mathbf{A}_k\mathbf{y} = \max_{i \in \mathbb{N}_m} \max_{j \in \mathbb{N}_n} a_{k,ij} \\ \underline{a}_k &= \min_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x}\mathbf{A}_k\mathbf{y} = \min_{i \in \mathbb{N}_m} \min_{j \in \mathbb{N}_n} a_{k,ij} \end{aligned}$$

The membership function of the fuzzy goals is of the shape



6. Player II's minimax solution

Theorem 0.3 *(Minimax solution). For a multi-objective two person matrix game with a linearly fuzzy goal functions, the Player II's minimax solution w.r.t. a degree of achievement of the fuzzy goals is equal to an optimal solution to the $r + 1$ constraints LP problem*

III-1 Multiobjective matrix game with fuzzy goals (2/2)

3. Player I's maximin solution

THEOREM 0.2 *(Maximin solution). For a multi-objective two person matrix game with a linearly fuzzy goal functions, the Player I's maximin solution w.r.t. a degree of achievement of the fuzzy goals is equal to an optimal solution to the $r+1$ constraints LP problem*

$$\begin{aligned} &\max_{x,\lambda} \lambda \\ &\textit{subject to} \\ &\frac{1}{\bar{a}_1 - \underline{a}_1}\left(\sum_{i=1}^{m} a_{1,ij}x_i - \underline{a}_1\right) \geq \lambda, (j \in \mathbb{N}_n) \\ &\frac{1}{\bar{a}_2 - \underline{a}_2}\left(\sum_{i=1}^{m} a_{2,ij}x_i - \underline{a}_2\right) \geq \lambda, (j \in \mathbb{N}_n) \\ &\cdots \\ &\frac{1}{\bar{a}_r - \underline{a}_r}\left(\sum_{i=1}^{m} a_{r,ij}x_i - \underline{a}_r\right) \geq \lambda, (j \in \mathbb{N}_n) \\ &\boldsymbol{e}'\boldsymbol{x} = 1,\ (\boldsymbol{e},\ \boldsymbol{x} \in \mathbb{R}^m), \\ &\boldsymbol{x} \geq 0. \end{aligned}$$

Proof : Nishizaki and Sakawa (2001), p. 48. □

4. Numerical example

In the Cook's example, three 3×3 payoff matrices are given by

$$A_1 = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & 7 & 2 \\ 0 & -2 & 0 \\ 3 & -1 & -6 \end{pmatrix}, A_3 = \begin{pmatrix} 8 & -2 & 3 \\ -5 & 6 & 0 \\ -3 & 1 & 6 \end{pmatrix}$$

We have $\bar{a}_1 = 6,\ \bar{a}_2 = 7,\ \bar{a}_3 = 8$ and $\underline{a}_1 = -2,\ \underline{a}_2 = -6,\ \underline{a}_3 = -5$. Then, we have to solve the LP problem

$$\begin{aligned} &\max_{\mathbf{x},\lambda} \lambda \\ &\text{subject to} \\ &2x_1 - x_2 + 2 \geq 8\lambda, \\ &5x_1 - 2x_2 + 3x_3 + 2 \geq 8\lambda, \\ &x_1 + 6x_2 - x_3 + 2 \geq 8\lambda, \\ &-3x_1 + 3x_3 + 6 \geq 13\lambda, \\ &7x_1 - 2x_2 - x_3 + 6 \geq 13\lambda, \\ &2x_1 - 6x_3 + 6 \geq 13\lambda, \\ &8x_1 - 5x_2 - 3x_3 + 5 \geq 13\lambda, \\ &-2x_1 + 6x_2 + x_3 + 5 \geq 13\lambda, \\ &3x_1 + 6x_3 + 5 \geq 13\lambda, \\ &\mathbf{e}.\mathbf{x} = 1,\ \mathbf{x} \geq 0. \end{aligned}$$

The optimal solution of Player I is $x_1 = 0.59928,\ x_2 = 0.15027,\ x_3 = 0.25045$ with a degree of satisfaction of 38.1 per cent.

$$\begin{aligned} &\min_{y,\lambda} \lambda \\ &\textit{subject to} \\ &\frac{1}{\bar{a}_1 - \underline{a}_1}\left(\sum_{j=1}^{n} a_{1,ij}y_j - \underline{a}_1\right) \leq \lambda, (i \in \mathbb{N}_m) \\ &\frac{1}{\bar{a}_2 - \underline{a}_2}\left(\sum_{j=1}^{n} a_{2,ij}y_j - \underline{a}_2\right) \leq \lambda, (i \in \mathbb{N}_m) \\ &\cdots \\ &\frac{1}{\bar{a}_r - \underline{a}_r}\left(\sum_{j=1}^{n} a_{r,ij}y_j - \underline{a}_r\right) \leq \lambda, (i \in \mathbb{N}_m) \\ &\boldsymbol{e}'\boldsymbol{y} = 1,\ (\boldsymbol{e},\ \boldsymbol{y} \in \mathbb{R}^n), \\ &\boldsymbol{y} \geq 0. \end{aligned}$$

7. Numerical example

In the Cook's example, three 3×3 payoff matrices is given by

$$A_1 = \begin{pmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & 7 & 2 \\ 0 & -2 & 0 \\ 3 & -1 & -6 \end{pmatrix}, A_3 = \begin{pmatrix} 8 & -2 & 3 \\ -5 & 6 & 0 \\ -3 & 1 & 6 \end{pmatrix}$$

We have $\bar{a}_1 = 6,\ \bar{a}_2 = 7,\ \bar{a}_3 = 8$ and $\underline{a}_1 = -2,\ \underline{a}_2 = -6,\ \underline{a}_3 = -5$. Then, we have to solve the LP problem

$$\begin{aligned} &\min_{\mathbf{y},\lambda} \lambda \\ &\text{subject to} \\ &2y_1 + 5y_2 + y_3 + 2 \leq 8\lambda, \\ &-y_1 - 2y_2 + 6y_3 + 2 \leq 8\lambda, \\ &3y_2 - y_3 + 2 \leq 8\lambda, \\ &-3y_1 + 7y_2 + 2y_3 + 6 \leq 13\lambda, \\ &-2y_2 + 6 \leq 13\lambda, \\ &3y_1 - y_2 - 6y_3 + 6 \leq 13\lambda, \\ &8y_1 - 2y_2 + 3y_3 + 5 \leq 13\lambda, \\ &-5y_1 + 6y_2 + 5 \leq 13\lambda, \\ &-3y_1 + y_2 + 6y_3 + 5 \leq 13\lambda, \\ &\mathbf{e}.\mathbf{y} = 1,\ \mathbf{y} \geq 0. \end{aligned}$$

The optimal solution of Player II is $y_1 = 0.38462,\ y_2 = 0.38462,\ y_3 = 0.23077$ with a degree of satisfaction of 61.5 per cent.

III.2 Multiobjective matrix game with fuzzy payoffs and fuzzy goals (1/3)

Let the multi-objective matrix game $G = (S^m, S^n, \bar{\mathbf{A}}_1, \ldots, \bar{\mathbf{A}}_r)$ with fuzzy goals and fuzzy constraints, where S^m and S^n denote the compact convex strategy spaces of Players, such that $S^m = \{\mathbf{x} \in \mathbb{R}^m_+, \mathbf{e}'\mathbf{x} = 1\}$ and $S^n = \{\mathbf{y} \in \mathbb{R}^n_+, \mathbf{e}'\mathbf{y} = 1\}$, and where the $\bar{\mathbf{A}}_k$'s $\in \mathbb{R}^{m \times n}$ are the payoff matrices of the game with fuzzy entries $\bar{a}_{ij} \in \mathfrak{F}(\mathbb{R})$.

1. Characterization of the fuzzy expected payoff

Let the fuzzy payoffs for each objective $k \in \mathbb{N}_r$ have the following LR-representation $\bar{a}_{k,ij} = (a_{k,ij}, a^-_{k,ij}, a^+_{k,ij})_{LR}$. The mean value is $a_{k,ij}$ and $a^-_{k,ij}$, $a^+_{k,ij}$ are the left and right spreads, respectively. Using mixed strategies, a fuzzy payoff is extended to a fuzzy expected payoff.

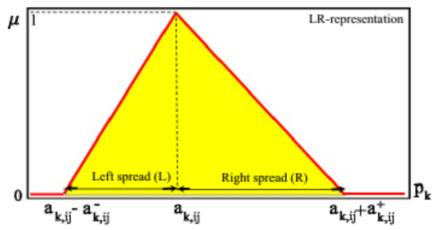
Definition 0.1 *(Fuzzy expected payoff). For any pair of mixed strategies $(\boldsymbol{x}, \boldsymbol{y})$, the k-th fuzzy expected payoff of Player I is*

$$\boldsymbol{x}\bar{A}_k\boldsymbol{y} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{k,ij}x_iy_j, \sum_{i=1}^{m}\sum_{j=1}^{n} a^-_{k,ij}x_iy_j, \sum_{i=1}^{m}\sum_{j=1}^{n} a^-_{k,ij}x_iy_j\right)_{LR}.$$

The MF is such that

$$\mu_{x\bar{A}y} : D_k \mapsto [0,1],$$

where D_k is the domain of the k-th payoff for Player I.

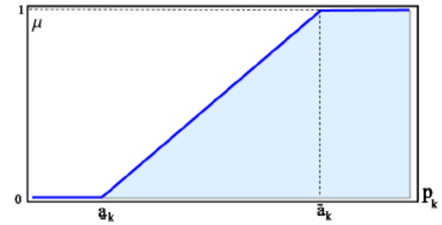


2. Fuzzy goals of the Player I

The Player is supposed to have a fuzzy goal for each of the r objectives. For any pair of mixed strategies $(\mathbf{x}, \mathbf{y})$ a membership function is denoted by $\mu_{\bar{G}_k}$ for the k-th payoff p_k. Assume that the degree of satisfaction increases linearly, we have

$$\mu\left(\bar{G}_k(p_k)\right) = \begin{cases} 1, & p_k > \bar{a}_k \\ \frac{p_k - \underline{a}_k}{\bar{a}_k - \underline{a}_k}, & \underline{a}_k \le p_k \le \bar{a}_k \\ 0, & p_k < \underline{a}_k, \end{cases}$$

where $\bar{a}_k$ and $\underline{a}_k$ are the best and the worst degree of satisfaction to the Player I, respectively. The membership function of the fuzzy goals is of the shape



3. Fuzzy payoffs

The MFs of the payoffs are expressed as

$$\mu(\bar{a}_{k,ij}(p_k) = \begin{cases} \frac{p_k - (a_{k,ij} - a^-_{k,ij})}{a^-_{k,ij}}, & p_k \in [a_{k,ij} - a^-_{k,ij}, a_{k,ij}) \\ \frac{a_{k,ij} + a^+_{k,ij} - p_k}{a^+_{k,ij}}, & p_k \in [a_{k,ij}, a_{k,ij} + a^+_{k,ij}] \\ 0, & p_k \ \texttt{not} \in [a_{k,ij} - a^-_{k,ij}, a_{k,ij} - a^+_{k,ij} \end{cases}$$

4. Extension Principle

Let a Cartesian product of universes be $X = X_1 \times \ldots \times X_r$ and r fuzzy sets $\bar{A}_1, \ldots, \bar{A}_r$ defined on $X_1, X_2, \ldots, X_r$ respectively. Let f be a mapping from X to the universe Y, such that $y = f(x_1, x_2, \ldots, x_r)$.

Definition 0.2 *The extension principle (Zadeh, 1965,1975) allows to define a fuzzy set $\bar{B}$ on Y trough f from the $\bar{A}_k$' s $(k \in \mathbb{N}_r)$ such that*

$$B = \left\{(y, \mu_{\bar{B}}(y)) | y = f(x_1, x_2, \ldots, x_r), (x_1, x_2, \ldots, x_r) \in X\right\},$$

where

$$\mu_{\bar{B}}(y) = \begin{cases} \sup_{(x_1,\ldots,x_r)\in X} \min\left\{\mu_{\bar{A}_1}(x_1), \ldots, \mu_{\bar{A}_r}(x_r)\right\}, & f^{-1} \neq \varnothing \\ 0, & \texttt{otherwise} \end{cases}$$

Example 1: Let $\bar{A}$, $\bar{B}$ be two fuzzy numbers. Using the extension principle the four basic arithmetic operations $* \in \{+, -, ., /\}$ on real numbers are extended to FNs by the expression

$$\mu_{(\bar{A}*\bar{B})}(z) = \sup_{z=x*y} \min\{\mu_{\bar{A}}(x), \mu_{\bar{B}}(y)\}$$

Example 2: For ordering r FNs $\bar{A}_1, \ldots, \bar{A}_r$, we consider a priority set P on $\{\bar{A}_1, \ldots, \bar{A}_r\}$, such ad $P\left(\bar{A}_k\right)$ is the degree to which $\bar{A}_k$ is ranked as the greatest FN. Using the extension principle, P is defined for each $k \in \mathbb{N}_r$ by the expression

$$P(\bar{A}_k) = \sup \min_{i \in \mathbb{N}_r} \bar{A}_i(u_i),$$

III.2 Multiobjective matrix game with fuzzy payoffs and fuzzy goals (2/3)

where the supremum is taken over all $(u_1, \dots, u_r) \in \mathbb{R}^r$ such that $u_k \geq u_j$ for all $j \in \mathbb{N}_r$.

Using the extension principle the MF of the k-th expected payoff $\mathbf{x}\mathbf{A}_k\mathbf{y}$ can be represented by the expression

$$\mu_{\mathbf{x}\mathbf{A}_k\mathbf{y}}(p) = \sup_{p=\mathbf{x}\mathbf{P}\mathbf{y}} \min \mu_{\tilde{a}_{k,ij}}(p_{ij}),\ P \in \mathbb{R}^{m\times n}$$

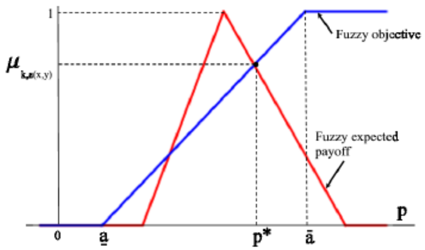
5. Degree of achievement of the aggregated fuzzy goal

Definition 0.3 *(Degree of achievement of a fuzzy goal). For any pair of mixed strategies $(\boldsymbol{x}, \boldsymbol{y})$, let the k-th expected payoff be $\mathbf{x}\tilde{\mathbf{A}}_k\mathbf{y}$ and k-th fuzzy goal be $\tilde{G}_k$. An achievement state of the the fuzzy goal is expressed by the intersection of the fuzzy expected payoff and goal. The MF of the fuzzy set is*

$$\mu_{k,a(x,y)}(p) = \min\left\{\mu_{x\tilde{A}_k y}(p), \mu_{\tilde{G}_k}(p)\right\},$$

where $p \in D_k$ is a payoff of Player I. A degree of achievement of the k-th fuzzy goal is defined as

$$\hat{\mu}_{k,a(\boldsymbol{x},\boldsymbol{y})}(p^*) = \max_{p} \mu_{k,a(\boldsymbol{x},\boldsymbol{y})}$$



The MF of the aggregated fuzzy goal is $\hat{\mu}_{k,a(x,y)}(p^*)$.

6. Player I's maximin solution

Definition 0.4 *(Maximin solution w.r.t. a degree of achievement of the aggregated fuzzy goal). For any pair of mixed strategies $(\boldsymbol{x}, \boldsymbol{y})$, given a degree of achievement of the aggregated fuzzy goal $\hat{\mu}_{k,a(\boldsymbol{x},\boldsymbol{y})}(p^*)$, the Player I's maximin value w.r.t. a degree of achievement of the aggregated fuzzy goal is*

$$\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \hat{\mu}_{k,a(\boldsymbol{x},\boldsymbol{y})}(p^*).$$

Then we have the expression

$$\max_{x} \min_{y} \min_{k\in\mathbb{N}_r} \max_{p_k} \min\left\{\mu_{\mathbf{x}\tilde{A}_k\mathbf{y}}(p_k), \mu_{\tilde{G}_k}(p_k)\right\}.$$

Theorem 0.5 *(Player I's maximin solution). For multiobjective two-person matrix games, with linear MFs of the fuzzy goals, and linear shape functions of FNs, the Player I's maximin solution w.r.t. a degree of achievement of the aggregated fuzzy goal is equal to an optimal solution to the nonlinear programming problem*

$$\max_{\boldsymbol{x},\sigma} \sigma$$

subject to

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{k,ij} + a^{+}_{k,ij})x_i y_j - \underline{a}_k}{\sum_{i=1}^{m}\sum_{j=1}^{n}(a^{+}_{k,ij}x_i y_j + \bar{a}_k - \underline{a}_k} \geq \sigma,\ \forall y \in \mathbb{R}^n,\ k \in \mathbb{N}_r$$

$$\boldsymbol{e}'\boldsymbol{x} = 1,\ (\boldsymbol{e}, \boldsymbol{x} \in \mathbb{R}^m),$$

$$\boldsymbol{x} \geq 0.$$

Proof : Nishizaki and Sakawa (2001), p. 65. □

For solving the problem, the algorithm consists in different steps : 1) the use of the relaxation procedure due to Shimizu and Aiyoshi (1980) by taking L points $\mathbf{y}^l, l \in \mathbb{N}_L$ satisfying $\mathbf{e}'\mathbf{y}_l = 1,\ y_l \geq 0$ and obtaining the optimal solution (x^L, σ^L); 2) r minimization problems have then to be solved. The variable transformation by Charnes and Cooper (1962) used for such linear fractional programming problem, induces LP problems.

7. Numerical example

Formulation

A multiobjective two-person matrix game (from Cook, 1976)with fuzzy payoffs and fuzzy goals is analyzed in Nishizaki and Sakawa (2001, p.70). Each Player has three pure strategies. The LR-representation of the fuzzy payoffs is

$$\tilde{\mathbf{A}}_1 = \begin{pmatrix} (2, .2, .2) & (5.5, .5) & (1, .8, .8) \\ (-1, .8, .8) & (-2, .4, .4) & (6, .1, .1) \\ (0, .1, .1) & (3, .5, .5) & (-1, .8, .8) \end{pmatrix},$$

$$\tilde{\mathbf{A}}_2 = \begin{pmatrix} (-3, .8, .8) & (7.3, .3) & (2, .4, .4) \\ (0, .5, .5) & (-2, .2, .2) & (0, .7, .7) \\ (3, .4, .4) & (-1, .8, .8) & (-6, .5, .5) \end{pmatrix},$$

$$\tilde{\mathbf{A}}_3 = \begin{pmatrix} (8, .1, .1) & (-2, .5, .5) & (3, .7, .7) \\ (-5, .5, .5) & (6, .4, .4) & (0, .6, .6) \\ (-3, .8, .8) & (1, .6, .6) & (6, .1, .1) \end{pmatrix}.$$

The fuzzy goals $\tilde{G}_1,\ \tilde{G}_2,\ \tilde{G}_3$ for the three objectives of Player I are represented by linear MFs

$$\mu_{\tilde{G}_1}(p_1) = \begin{cases} 1,\ p_1 > 6.5 \\ (p_1+1)/7.5,\ -1 \leq p_1 \leq 6.5 \\ 0,\ p_1 \leq -1 \end{cases}$$

$$\mu_{\tilde{G}_1}(p_2) = \begin{cases} 1,\ p_2 > 5.5 \\ (p_2+2)/7.5,\ -1 \leq p_2 \leq 6.5 \\ 0,\ p_2 \leq -2 \end{cases}$$

$$\mu_{\tilde{G}_1}(p_3) = \begin{cases} 1,\ p_3 > 5.8 \\ (p_3+1)/6.8,\ -1 \leq p_3 \leq 5.8 \\ 0,\ p_3 \leq -1 \end{cases}$$

III.2 Multiobjective matrix game with fuzzy payoffs and fuzzy goals (3/3)

Shimizu and Aiyoshi 's iterative method

The solution for Player I is obtained after three iterations. We have

$$x_1 = .4434,\ x_2 = .3178,\ x_3 = .2388.$$

Sakawa's direct method : Player I

Using properties of some constraints, the initial nonlinear programming problem for Player I, Sakawa (1983) retains the following equivalent programming problem

$$\max_{\mathbf{x},\sigma}\ \sigma$$

subject to

$$\frac{\sum_{i=1}^{m}(a_{k,ij}+a^{+}_{k,ij})x_i-\underline{a}_k}{\sum_{i=1}^{m}a^{+}_{k,ij}x_i+\bar{a}_k-\underline{a}_k}\geq\sigma,\ j\in\mathbb{N}_n,\ k\in\mathbb{N}_r$$

$$\mathbf{e}'\mathbf{x}=1,\ (\mathbf{e},\ \mathbf{x}\in\mathbb{R}^m)$$

$$\mathbf{x}\geq 0.$$

The results for Player I are obtained with a degree of satisfaction of 24.6 per cent.

```
NMaximize[{σ,
  ((B1[[1]] + F1[[1]]).x - anl1) / (F1[[1]].x + ab1 - anl1) ≥ σ,
  ((B1[[2]] + F1[[2]]).x - anl1) / (F1[[2]].x + ab1 - anl1) ≥ σ,
  ((B1[[3]] + F1[[3]]).x - anl1) / (F1[[3]].x + ab1 - anl1) ≥ σ,
  ((B2[[1]] + F2[[1]]).x - anl2) / (F2[[1]].x + ab2 - anl2) ≥ σ,
  ((B2[[2]] + F2[[2]]).x - anl2) / (F2[[2]].x + ab2 - anl2) ≥ σ,
  ((B2[[3]] + F2[[3]]).x - anl2) / (F2[[3]].x + ab2 - anl2) ≥ σ,
  ((B3[[1]] + F3[[1]]).x - anl3) / (F3[[1]].x + ab3 - anl3) ≥ σ,
  ((B3[[2]] + F3[[2]]).x - anl3) / (F3[[2]].x + ab3 - anl3) ≥ σ,
  ((B3[[3]] + F3[[3]]).x - anl3) / (F3[[3]].x + ab3 - anl3) ≥ σ,
  x1 + x2 + x3 == 1, x1 ≥ 0, x2 ≥ 0, x3 ≥ 0}, {x1, x2, x3, σ}]
```

```
{0.246064, {x1 → 0.443378, x2 → 0.317832, x3 → 0.23879, σ → 0.246064}}
```

Sakawa's direct method : Player II

Using properties of some constraints, the initial nonlinear programming problem for Player II, Sakawa (1983) retains the following equivalent programming problem

$$\max_{\mathbf{y},\lambda}\ \lambda$$

subject to

$$\frac{\sum_{j=1}^{n}(a_{k,ij}+a^{+}_{k,ij})y_j-\underline{a}_k}{\sum_{j=1}^{n}a^{+}_{k,ij}y_j+\bar{a}_k-\underline{a}_k}\leq\lambda,\ i\in\mathbb{N}_m,\ k\in\mathbb{N}_r$$

$$\mathbf{e}'\mathbf{y}=1,\ (\mathbf{e},\ \mathbf{y}\in\mathbb{R}^n)$$

$$\mathbf{y}\geq 0.$$

The results for Player II are obtained with a degree of satisfaction of 58.5 per cent.

```
NMinimize[{λ,
  ((A1[[1]] + E1[[1]]).y - anl1) / (E1[[1]].y + ab1 - anl1) ≤ λ,
  ((A1[[2]] + E1[[2]]).y - anl1) / (E1[[2]].y + ab1 - anl1) ≤ λ,
  ((A1[[3]] + E1[[3]]).y - anl1) / (E1[[3]].y + ab1 - anl1) ≤ λ,
  ((A2[[1]] + E2[[1]]).y - anl2) / (E2[[1]].y + ab2 - anl2) ≤ λ,
  ((A2[[2]] + E2[[2]]).y - anl2) / (E2[[2]].y + ab2 - anl2) ≤ λ,
  ((A2[[3]] + E2[[3]]).y - anl2) / (E2[[3]].y + ab2 - anl2) ≤ λ,
  ((A3[[1]] + E3[[1]]).y - anl3) / (E3[[1]].y + ab3 - anl3) ≤ λ,
  ((A3[[2]] + E3[[2]]).y - anl3) / (E3[[2]].y + ab3 - anl3) ≤ λ,
  ((A3[[3]] + E3[[3]]).y - anl3) / (E3[[3]].y + ab3 - anl3) ≤ λ,
  y1 + y2 + y3 == 1, y1 ≥ 0, y2 ≥ 0, y3 ≥ 0}, {y1, y2, y3, λ}]
```

```
NMinimize::incst : NMinimize was unable to generate any initial points satisfying the inequality constraints
  {-1. + 1. y2 + 1. y3 ≤ 0, «1»/«1» - λ ≤ 0, «6», «1» - λ ≤ 0, (1 - 1.6 y2 - 0.2 (1. - «1» - 1. y3) + 6.1 y3)/(7.5 + 0.4 y2 + 0.8 (1. + Times[«2»] + Times[«2»]) + 0.1 y3) - λ ≤ 0}. The initial
  region specified may not contain any feasible points. Changing the initial region or specifying explicit initial points may provide a better solution. ≫
```

```
{0.585032, {y1 → 0.415622, y2 → 0.449909, y3 → 0.134469, λ → 0.585032}}
```

Thank You for attention !