INTERNATIONAL CONFERENCE

GAME THEORY AND MANAGEMENT

GTM2007

Stochastic Differential Games and Queueing Models To Innovation and Patenting

by André A. Keller, Université de Haute Alsace - France

Group for Learning, Innovation and Knowledge in Organizations

andre.keller@uha.fr

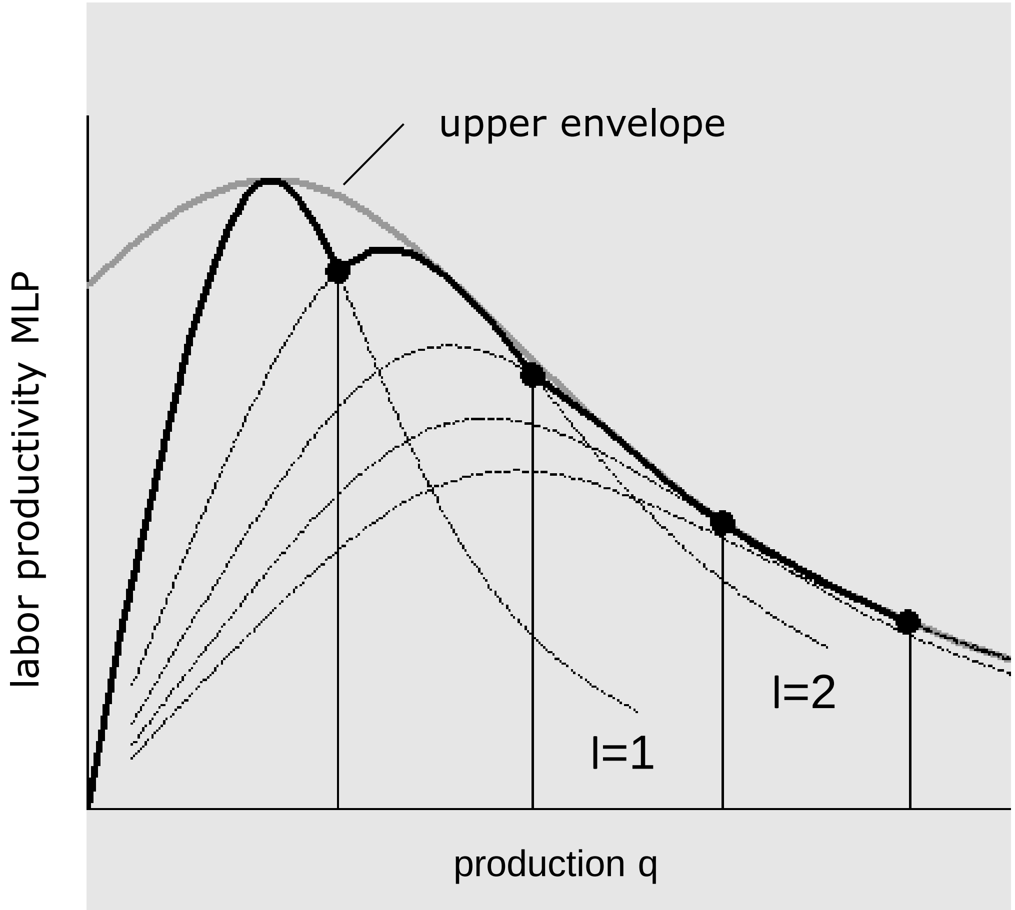
Organized by

SCHOOL OF MANAGEMENT (SOM),
ST. PETERSBURG STATE UNIVERSITY (SPbU)
THE INTERNATIONAL SOCIETY OF DYNAMIC GAMES (ISDG)
(Russian Chapter)

Contents

- 1. Introduction to stochastics
  (**1.1** piecewise functions and ODEs in economics; **1.2** piecewise deterministic processes; **1.3** Itô's lemma for Poisson processes, **1.4** a simple TFP stochastic model)
- 2. Optimal control with jumps in the state variables
  (**2.1** optimal control with upwards jumps in the state variables; **2.2** optimal interior solution; **2.3** application to a fictive patenting example; **2.4** resolution of the problem with one jump; **2.5** interior upwards jump solution)
- 3. A stochastic game of R&D competition
  (**3.1** innovation queueing game; **3.2** stochastic innovation game, **3.3** characteristics of the game; **3.4** solving the innovation game)

1.1a Piecewise functions in Economics



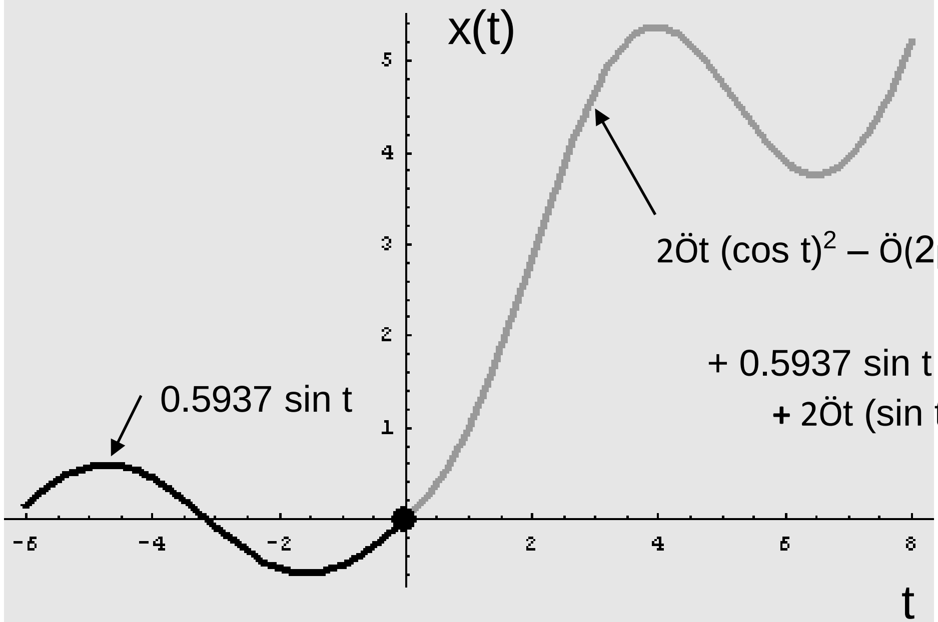
$TC(q)=2q-2q^2+q^3+(q^4/l^3)+3l$
$TC=w\ L$
$MLP=q/L$
$w=100$

q : production
MLP : mean labor productivity
TC : total cost
w : wages
l : size of equipment

<u>Note</u> : no fixed cost, no other variable cost, constant wages

1.1b Piecewise ODEs

$$x''(t) + x(t) = \begin{cases} 0, & -\infty < t \le 0 \\ 2\sqrt{t}, & \text{otherwise} \end{cases}$$

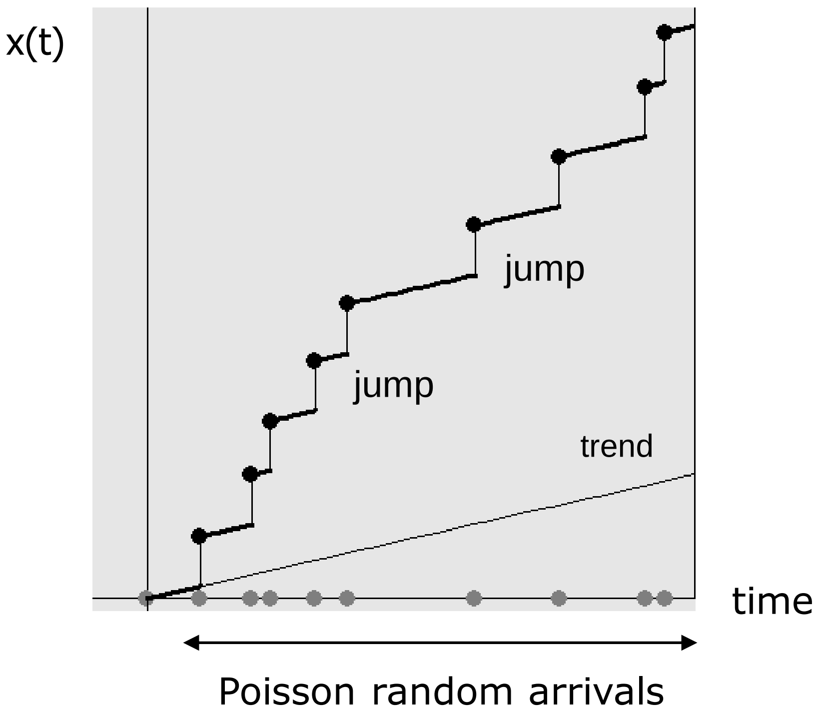


$$2\sqrt{t}\,(\cos t)^2 - \sqrt{2\pi}\,\cos t \int_0^{\sqrt{2t/\pi}} \cos(\pi s^2/2)\,ds$$

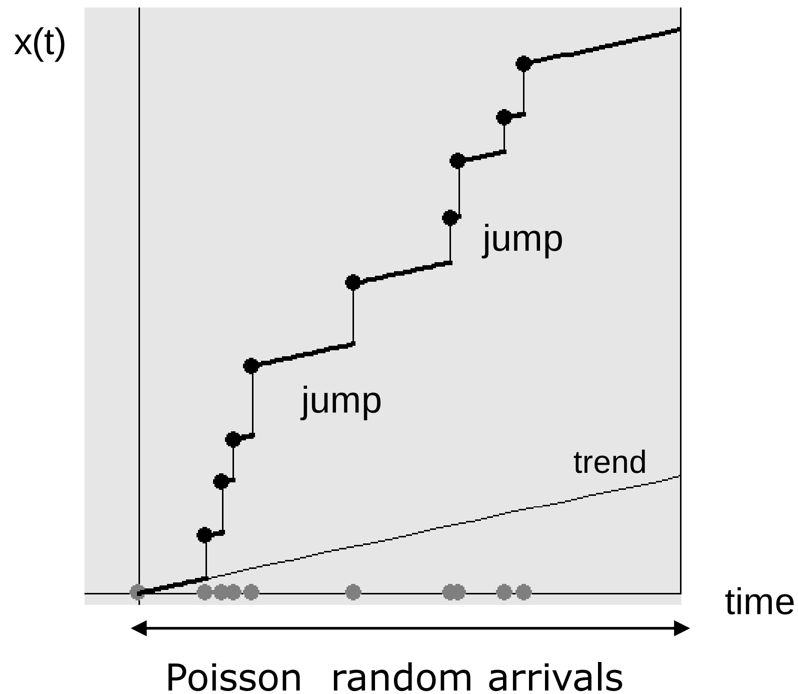
$$+\ 0.5937 \sin t - \sqrt{2\pi}\,\sin t \int_0^{\sqrt{2t/\pi}} \sin(\pi s^2/2)\,ds + 2\sqrt{t}\,(\sin t)^2$$

1.2 Piecewise deterministic processes

constant jump amplitude



Normal random jump amplitude



The SDE of the Poisson process is defined by **dx(t) = a dt + b dq(t)** where the increment **dq(t)** is driven by

$$dq(t) = \begin{cases} 0, & \text{w.p. } 1\text{-}l\, dt \\ 1, & \text{w.p. } l\, dt \end{cases}$$

1.3 Itô's lemma for Poisson processes

Lemma (1-dimentional Itô formula for Poisson processes). Let x(t) be an Itô process given by a SDE with constants a and b

$$dx(t) = a\,dt + b\,dq(t),$$

where the increment q(t)-q(t) in any interval of length |t-t| is Poisson distributed $\mathcal{P}$oi(l(t-t)) with mean l(t-t).

Let F(t,x(t)) a twice continuous differentiable function $\mathcal{C}^2$([0, ¥) x R).. Then we have the Itô's formula

$$dF(t,x(t)) = (F_t + a\,F_x)dt + \{F(t,x(t)+b) - F(t,x(t))\}\,dq(t)$$

1.4 Simple TFP stochastic model

Let the total factor productivity (TFP) A(t) have a SDE with constants g and s

$$dA(t)/A(t) = g\,dt + s\,dq(t).$$

Applying Itô' formula, we take F(t,A(t)) = ln A(t). Then we have $F_t = 0$, $F_A = A^{-1}(t)$ and ln(A(t) + s A(t)) –ln A(t) = ln(1+s). According to the Itô's formula we have

$$d \ln A(t) = g + \ln(1+s)\, dq(t)$$

Then by integrating both sides and letting $A(0) = A_0$, we have

$$A(t) = A_0\, e^{g\,t + \ln(1+s)\,\{q(t)-q(0)\}}$$

2.1 Optimal control with upward jumps in the state variables

Problem $\mathcal{P}_1$: maximizing a zero-discounted functional subject to J optimally or randomly upward jumps, inside the time interval [0,T]. Find control u(t), jumps dates T_j , value of the states at jumps $x(T_j^-)$ and $x(T_j^+)$

$$\max \int F(t, x(t),u(t))\, dt + \sum_j p(T_j)(x(T_j^-)-x(T_j^+))$$

s.t.

$x'(t) = f(t,x(t),u(t))$, except at T_j ,j=1,…,J,

$$x(T_j^+) \geq x(T_j^-)$$

$x(0)=x_0$, $x(T)=x_T$ given

Notation :

T_j, j=1,…, J : dates of jumps ; $x(T_j^-)$ value of the state variable immediately before the jump; $x(T_j^+)$ value of the state variable immediately after the jump; $p(T_j)$ price for an extra unit of the state variable x at jump T_j

2.2 Optimal interior solution

The Hamiltonian is given by

$$\mathcal{H}(t,x(.),p(.),u(.)) = F(t,x(t),u(t)) + p(t)\, f(t,x(t),u(t))$$

<u>Theorem</u> : An optimal solution of the optimal control problem $\mathcal{P}_1$ with upward jumps in the state variable must satisfy the necessary conditions

(i)	$\mathcal{H}_u(.)$	$=$	$F_u(.) + p(t)\, f_u(.) = 0$,
(ii)	$x(T_j^+)$	$\geq$	$x(T_j^-)$,
(iii)	$x'(t)$	$=$	$\mathcal{H}_p(.) = f(t,x(t),u(t))$, <u>except at</u> T_j, $j=1,\ldots,J$,
(iv)	$p'(t)$	$=$	$-\mathcal{H}_x(.) = -F_x(.) - p\, f_x(.)$,
(v)	$p(t)$	$\geq$	$p(t)$ <u>for all</u> $t \in [0,T]$,
(vi)	$p(T_j)$	$=$	$p(T_j)$, $j = 1,\ldots, J$,

(vii) $\mathcal{H}(T_j^-) - \mathcal{H}(T_j^+) + p'(T_j)(x(T_j^-) - x(T_j^+)) = 0$ <u>if</u> $T_j \in (0,T)$, $j=1,\ldots, J$

2.3 Application to a fictive patenting model

Problem : A firm has a stock of patents $x(0)=2$ for a unique good. The rate of producing $u(t)$ is governed by the ODE $x'(t) = -u(t)$. The patents have no cost. The patent allow for producing a final product by using L fixed factor according to the production function $q = 2\sqrt{u}\sqrt{L}$. The output unit price is 1 and L=1. The firm can buy additional patents at $p(t)=(t-1)^2+1$ but do not sell any patents. The planning period is [0,2]. Suppose there is only one jump T_1 at date 1. The firm will find u such that

$$\text{maximize} \int_0^2 2\sqrt{u(t)}\,dt + p(T_1)(x(T_1^-) - x(T_1^+))$$

s.t.

$$x'(t) = -u(t),\ x(0)=2,\ x(2)=1,$$

$$x(T_1^+) \geq x(T_1^-)$$

2.4 Resolution of the problem with one jump

The Hamiltonian is $\mathcal{H}$(x(.), p(.), u(.)) = 2 Öu(t) – p (t) u(t). The FOCs are given by *(i)* $\mathcal{H}_u$= 1/Öu- p = 0, *(ii)* x′(t) = $\mathcal{H}_p$ = - u, *(iii)* p′(t) =-$\mathcal{H}_x$ = 0 *(iV)* x_0 = 2, x_T = 1. We deduce that u(t) **= u**, **p(t) = p** and dx(t) **= -** u dt**.** Hence, u(t) = ½, p(t) =Ö2, x(t) = 2 -1/2 t.

0 £ t < 1 (to the left)

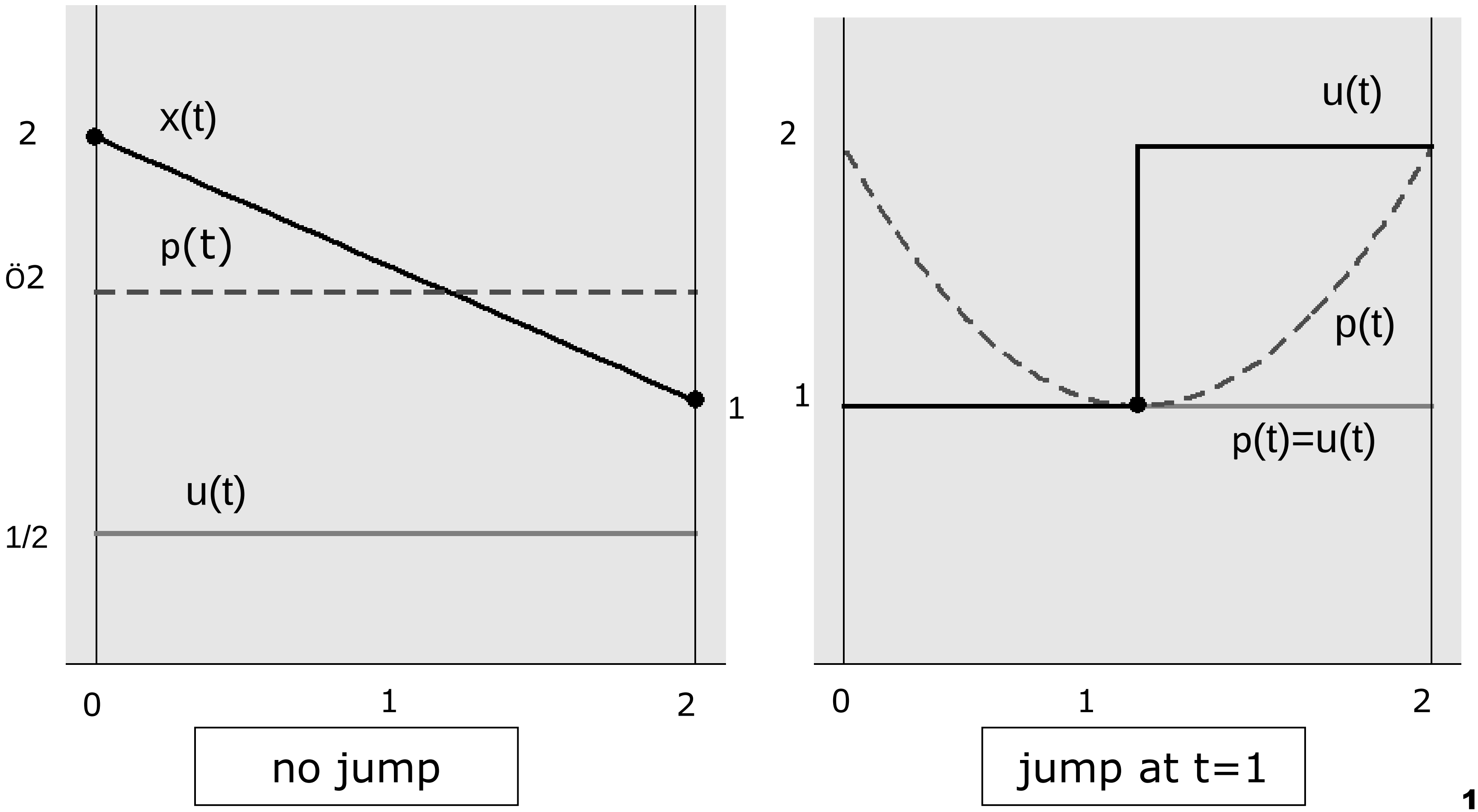
{u(t)=1, x(0)=2}
þ{**ODE**: x′(t)=-1,x(0)=2},
p(t)=1/Öu(t)=1,
lim $_{t®1^-}$ x(t)= **lim** $_{e®0}$ {2 –(1-e)}= 1

1< t £ 2 (to the right)

{u(t)=1, x(2)=1}
þ{**ODE**: x′(t)=-1,x(2)=1},
p(t)=1/Öu(t)=1,
lim $_{t®1^+}$ x(t)= **lim** $_{e®0}$ {3 –(1+e)}= 2

The optimality condition $\mathcal{H}(1^-)$- $\mathcal{H}(1^+)$+ p′(1) (x(1^-)-x(1^+)) = 0 is satisfied.

2.5 Interior upward jump solution



3.1a Innovation Queueing Game :

presentation

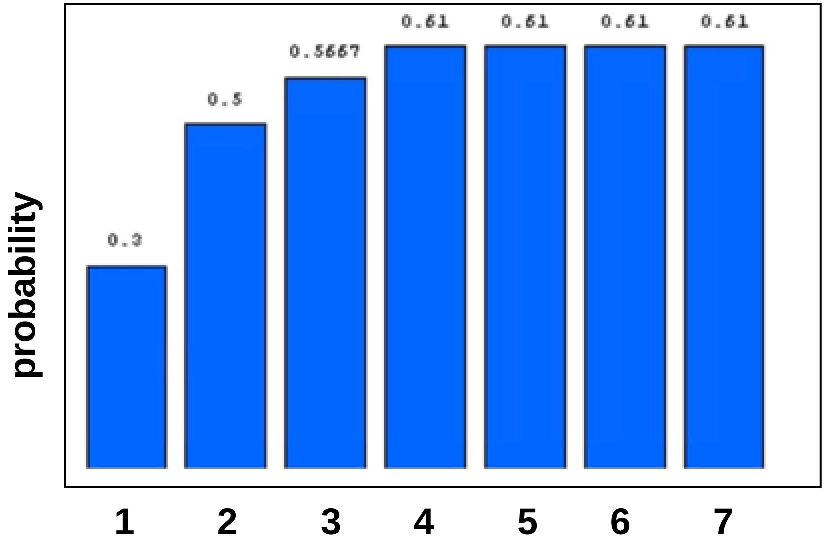
- Each firm of an industry spends the same amount of $ 1million R&D
- The produced and patented innovation yields an amount of $ 10 millions to the winner
- The probability that one innovation is successfully developed depends on the total amount invested by the industry.
- The more the industry will invest, the greater the probability of success will be, with diminishing returns.

3.1b Innovation Queueing Game :

How much would be invested in the industry ?

Efficient number of firms developing this innovation

R&D investment of industry (in million $)



in millions $

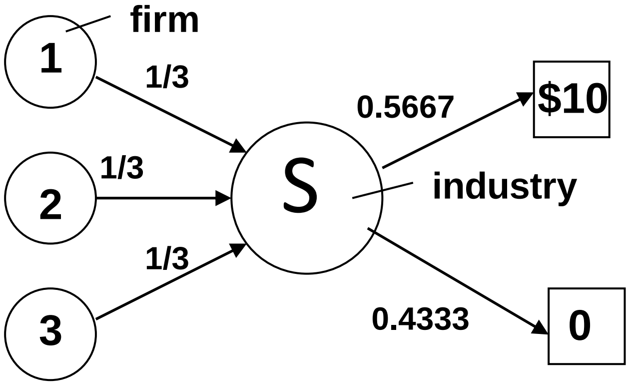
R&D	ER	MR	MC
1	**3.0**	**3.0**	**1.0**
2	**5.0**	**2.0**	**1.0**
3	**5.667**	**0.667**	**1.0**
4	**6.1**	**0.433**	**1.0**
5	**6.1**	**0.**	**1.0**
6	**6.1**	**0.**	**1.0**
7	**6.1**	**0.**	**1.0**

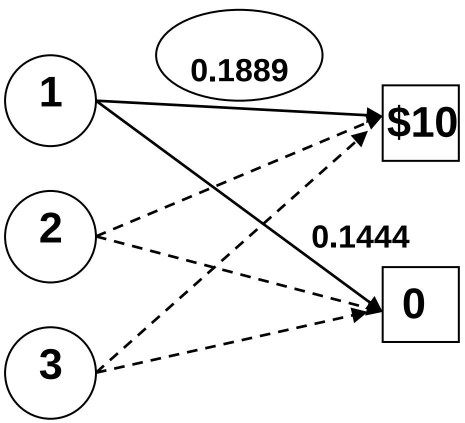
ER : expected revenu
MR : marginal revenu
MC : marginal cost

MR<MC

3.1c Innovation Queueing Game :

how many firms will attend to the innovation race ?





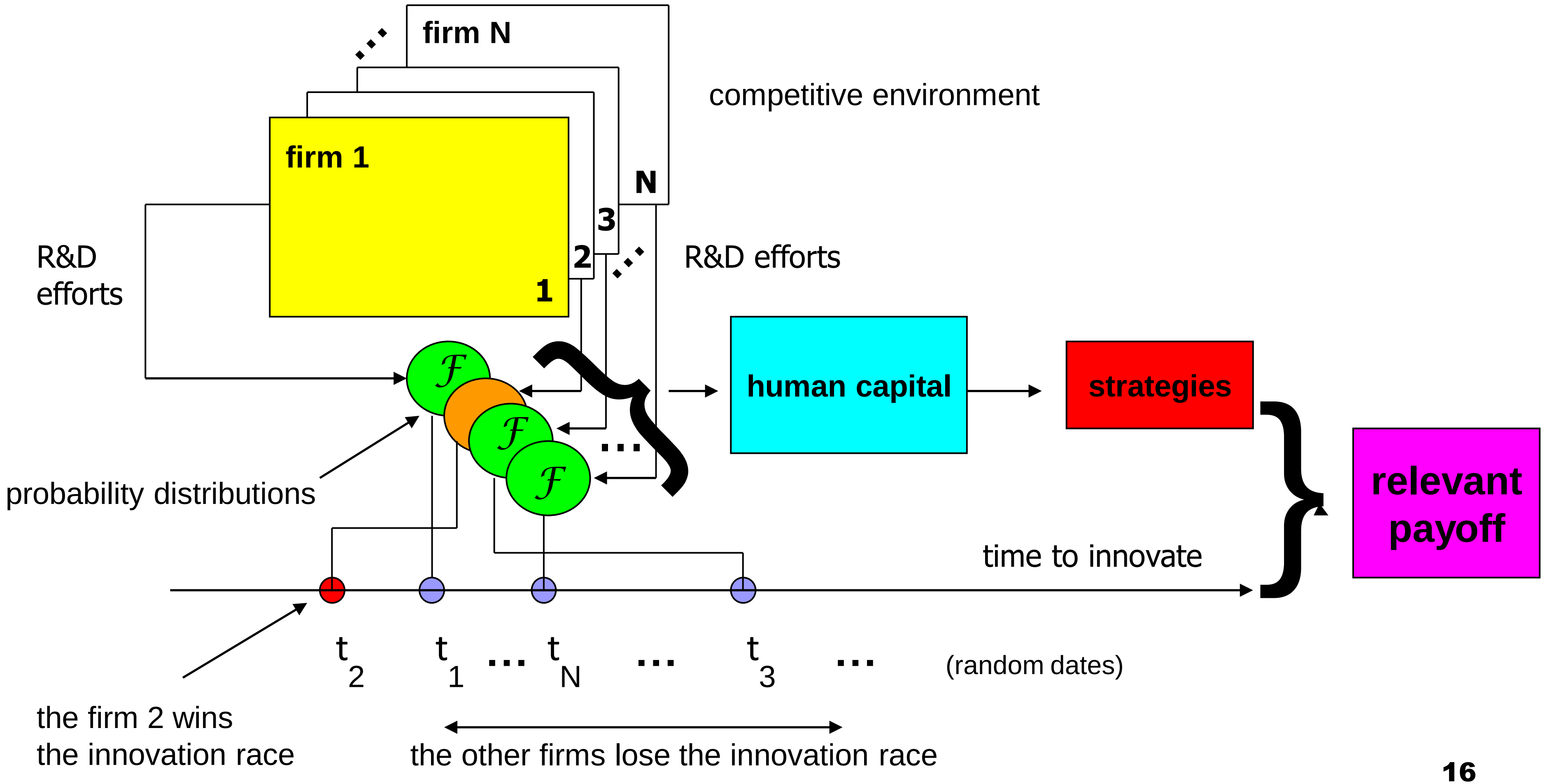
number of firms attending the race

RD	Pr	ER	MC	NB	Y/N
1	.3	3.	1	2.	Y
2	.25	2.5	1	1.5	Y
3	.1889	1.889	1	0.889	Y
4	.1525	1.525	1	0.525	Y
5	.122	1.220	1	0.220	Y
6	.10175	1.1017	1	0.1017	Y
7	.0871	0.871	1	-0.129	N

NB : net benefit; Pr : probability

3.2 Stochastic innovation game :

Loury (1979), Lee & Wilde (1980), Dockner et al. (2000)



3.3 Characteristics of the game

- Economic framework : N competitive firms try to introduce innovations (new products, technologies, services). The winning firm acquires a monopolistic position and the N-1 competitors are kept out of the market by a patent protection. The firms have competing R&D projects and no one knows the invested R&D of others. R&D expenses exert positive externalities, with higher accumulation of know-how.

- Technical assumptions : the times t_i to complete a project is random and i.i.d. with probability distributions $F_i = P\{t_i < t\}$. The date of innovation is min $\{t_i, i=1,\ldots,N\}$.. The rate F'_i is proportional to the R&D efforts. The cost of R&D efforts is quadratic in the investment rate.

- A stochastic differential game : this game belongs to the class of piecewise deterministic games with N+1 modes. Before the innovation occurs, the mode is zero. The system will then switch to mode i Î {1,2,...,N}. All players are supposed to maximize the expected discounted profit. The profit consists in 3 terms that are weighted by their probability : present value of the net benefits in case of succeeding the innovation race, present value of the net benefits in case of a lossing the race, and present value of R&D efforts. Taking the expectation leads to a deterministic control problem.

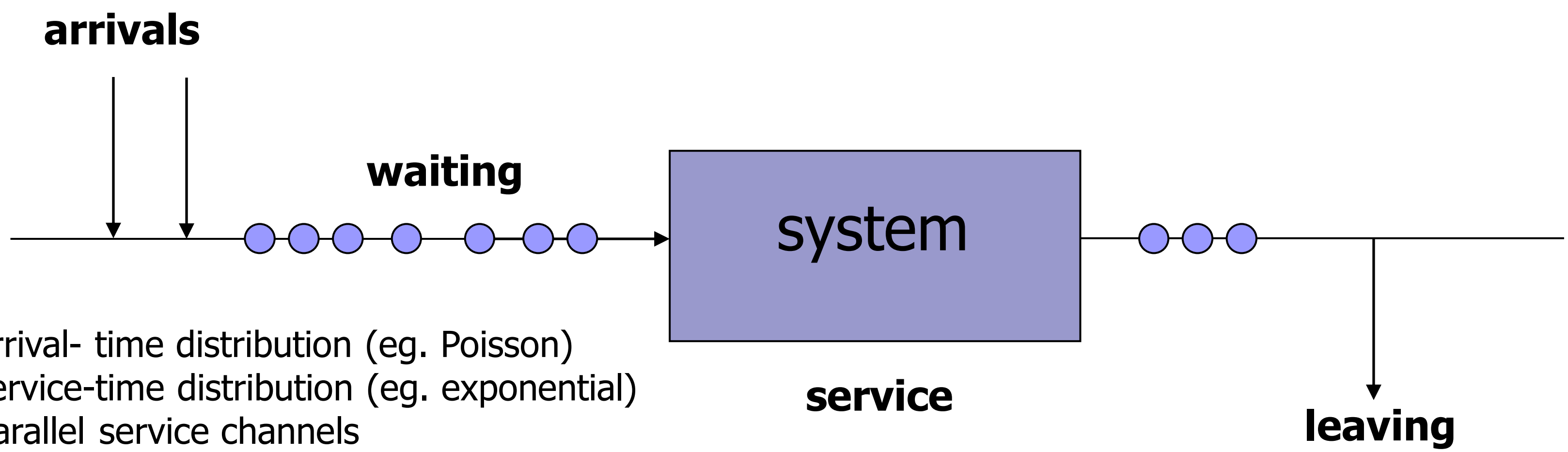
3.4 Solving the innovation game

- The distributions F_i's (i=1,…,N) are the state variables and the rates of R&D efforts u_i's (i=1,…,N) the controls. From the current value Hamiltonians the FOCs are deduced (N+1 boundery conditions). The game is transformed to an exponential game. In exponential games, the state variable do not enter the RHS of the system dynamics. However, this variable enters the objective function in an exponential way.
- All firms observe and base their strategies upon the N-dimentional vector z whose components are defined by $z'_i = u_i$. Let the aggregate stock of know-how be

  $y = e^{-l S z_j(t)}$. The players condition their strategies rather on y (state variable) than on $(z_1, z_2, \ldots, z_N)$.
- Let us consider a Nash equilibrium with open-loop strategies. The solutions of the state and costate equations are independently determined. The effort rates in R&D $u_i(t)$, i=1,…,N are independent of the industry-wide stock of know-how y. All firms have identical effort strategies. The open loop equilibrium is Markov perfect.

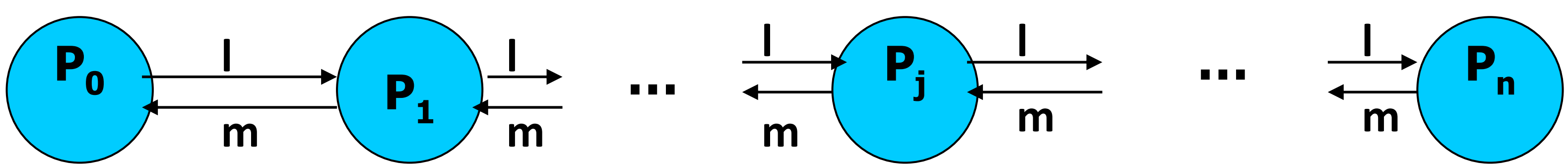
Thank You for your attention

3.1 Queue model A/B/X/Y/Z



A : arrival- time distribution (eg. Poisson)
B : service-time distribution (eg. exponential)
X : parallel service channels
Y : restriction on system capacity
Z : queue discipline (eg. FCFS)

M/M/1 : Poisson input (**l**)/exponential service (**m**)/single-server (**1**)



$P'_0(t) = -lP_0 + mP_1$, $P'_1(t) = lP_0 - lP_1 - mP_1 + mP_2$,…, $P'_n(t) = lP_{n-1} - lP_n - mP_n + mP_{n+1}$