Circuit Analysis to Environmental Economic Dynamics and Control

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Abstract: This study extends to environmental economics, the use of the conventional circuit analysis and control in engineering. Resources and environmental management problems are investigated with help of continuous-time systems: growth models with polluting events and fishery models with open-access to the industry. This study introduces the fundamental block-diagram approach, with small-size applications to pollution and fisheries management problems. The software *MATHEMATICA* (version 7) and its application packages are used for analyzing and solving the systems, symbolically and numerically.

Key–Words: Circuit analysis. Transfer function. BIBO stability. Lyapunov stability. EKC relationship. Green Solow-Swan model. Open-access fishery model.

1 Elementary circuit theory

The fundamental elements of circuit theory are notably from [1, 4, 8]. The computations are using the specialized MATHEMATICA packages: "Control Systems Professional", "Polynomial Control System" and "SchematicSolver".

1.1 Signals

Definitions. Signals are abstract or physical elements which convey information about the behavior of a system . The signals are often represented as a function of time. The variation of the signal value is the waveform. The signals are of two main types: the analog signals for continuous-time systems and the discrete signals for discrete-time systems. An analog signal is of the form $x(t)$ [1]. The fundamental analog and discrete basic signals are the unit impulse signal and Dirac distribution , the unitstep signal and Heaviside function , the unit ramp signal , unit sinusoidal signal , unit exponential signal. A sinusoidal input signal may be exponentially modulated, such as

$$u(t) = Ae^{\alpha t}\cos(\omega t + \varphi),$$

where A denotes the amplitude, $2\pi/\omega$ the period, $\omega = 2\pi f$ the angular frequency and φ the phase [2].

1.2 Phasor Representation

Definitions. Let a sinusoidal signal be defined by

$$x(t) = A\cos(\omega t + \phi),$$

where A is the magnitude, ω the angular frequency in radian/second and ϕ the phase angle in radian or in degree. The wave reaches a peak at $t = -\phi/\omega$, since we have $\omega t + \phi = 0$. The wave length is the period T and the frequency is defined by $f = 1/T$ and measured in Hz. The frequency f and the angular frequency ω are related by $\omega = 2\pi f$. The polar representation of a phasor $\underline{V} = A\angle\phi$ is encoding the amplitude and phase of a sinusoid and represents [3] the complex constant $Ae^{j\phi}$.

Phasor arithmetic. The sum of two phasors $\underline{V}_1$ and $\underline{V}_2$ is $A_1\angle\phi_1 + A_2\angle\phi_2$. The multiplication of a phasor $\underline{V} = A\angle\phi$ by a complex constant $Be^{j\theta}$ produces the phasor $(AB)\angle(\theta + \phi)$. The time differentiation of the sinusoidal $x(t) = A\cos(\omega t + \phi)$ is $x'(t) = -\omega A\sin(\omega t + \phi)$. The phasor of the derivative signal is $-\omega A\angle\phi - 90 = \omega A\angle\phi + 90°$. Differentiating a sinusoidal signal is equivalent to multiplying

[1] A discrete signal may be deduced from $x(t)$ by uniform sampling every T units of time, such as

$$x(kT) = x(t),\ t = 0, \pm T, \pm 2T, \ldots, \pm kT, \ldots,$$

where T denotes the time step (or sample interval) in seconds (s), and $f = \frac{1}{T}$, the sampling frequency (i.e. the number of samples per unit of time) in Hertz (Hz) or cycle/second.

[2] This expression is the real part of $Ae^{j\varphi}e^{st}$, where $s = \alpha + j\omega$ is a complex number.

[3] The phasor also refers to $Ae^{j\phi}e^{j\omega t}$. Indeed, we have

$$A\cos(\omega t + \phi) = \frac{A}{2}e^{j(\omega t+\phi)} + \frac{A}{2}e^{-j(\omega t+\phi)}.$$

The sine wave is then the real part of $Ae^{j(\omega t+\phi)}$.

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the phasor by $j\omega = e^{j\pi/2}$ (i.e a multiplication by ω and a rotation by 90°). Similarly, integrating a phasor corresponds to the multiplication by

$$(j\omega)^{-1} = \frac{e^{-j\pi/2}}{\omega}.$$

1.3 Linear Time-invariant (LTI) Systems

Definitions. LTI systems are represented by differential (or difference) equations with constant (or variable) coefficients, s.a. with the *n*th-order system

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{k=0}^{m} b_k x^{(k)}(t),\ n \geq m$$

where x denotes the system input, y the system output, $y^{(i)}(t)$ the generic *i*th time derivative of $y(t)$ such that $y^{(i)}(t) \equiv d^i y(t)/dt^i$ and $x^{(k)}(t)$ the generic *k*th time derivative of $x(t)$. Using the differential operators, we get the algebraic expression

$$\sum_{i=0}^{n} a_i D^i y(t) = \sum_{k=0}^{m} b_k D^k x(t). \tag{1}$$

For a known input $x(t)$ and given coefficients $a's$ and $b's$, a unique solution $y(t)$ is obtained for a set of the n initial conditions, about $y(t)$ and the $n-1$ derivatives :

$$y(0) = y_0, y'(0), y''(0), \ldots, y^{(n-1)}(0).$$

Total response decomposition. The total response may be divided into specific components according two different ways. Firstly, the total response is divided into a free response and a forced response: the free response or zero-input response only depends on the initial conditions; the forced response or zero-state response only depends on the input. Secondly, the total response is the sum of a transient response component and a steady-state response: the transient component will approach zero as time tends to infinity, while the steady-state component does not.

1.4 Transfer Function Representation

Definitions. Let a SISO (Single Input-Single Output) system be the linear constant-coefficients ODE (1). Applying the $\mathcal{L}$-transform to the signals, we get

$$\Big(\sum_{i=0}^{n} a_i s^i\Big) Y(s) = \Big(\sum_{k=0}^{m} b_k s^k\Big) X(s).$$

The system transforms the input signal $X(s)$ by the rational transfer function (TF) $H(s)$, s.a.

$$H(s) = \frac{B(s)}{A(s)},\ B(s) \equiv \sum_{k=0}^{m} b_k s^k,\ A(s) \equiv \sum_{i=0}^{n} a_i s^i.$$

The poles are the (real and complex) roots of the equation $A(s) = 0$. The zeros are the (real and complex) roots of the equation $B(s) = 0$. In factored form, the pole-zero representation is

$$H(s) = H_0 \frac{\prod_k (s - z_k)}{\prod_i (s - p_i)},$$

where H_0 denotes the scale factor, *z*'s and *p*'s are the zeros and poles, respectively [4].

Bode diagrams. To display the function $H(s)$ of the complex s, two plots are necessary: one is the magnitude plot for the amplitude $|H(s)|$, the other is the phase plot for the phase $\arg H(s)$.
Given a TF with a real pole of the form

$$H(s) = \frac{1}{\frac{s}{\omega_0} + 1},$$

where ω_0 denotes the break frequency. When $s = j\omega$, the phasor representation is

$$H(j\omega) = |H(s)| \angle H(j\omega).$$

The magnitude in dB is

$$|H(j\omega)| = -20 \log_{10} \sqrt{1 + (\frac{\omega}{\omega_0})^2}.$$

The low frequency asymptote is one horizontal. The high frequency asymptote is a straight line with slope of -20 dB/decade, through the break frequency ω_0 at 0 dB. The maximum error between the approximated linear piecewise magnitude and the true magnitude is approximately 3 dB ($-20 \log_{10} \sqrt{2} = -3.01$). Figure 1 shows the approximate magnitude at zero, when $\omega_0 = 10$, until the break frequency and then dropping at 20 dB/decade. The TF's phase with a single real pole is

$$\angle H(j\omega) = -\angle\big(1 + j\frac{\omega}{\omega_0}\big) = -\tan^{-1}\big(\frac{\omega}{\omega_0}\big).$$

At low frequencies ($\omega << \omega_0$), the approximated phase is 0 rad. At high frequencies ($\omega >> \omega_0$), the approximated phase is $-\frac{\pi}{2}$ $\texttt{rad}$. At the break frequency ($\omega = \omega_0$), the phase is $-\frac{\pi}{4}$ $\texttt{rad}$.

[4] The concept of partial transfer function (PTF) is adapted for multiple-input, multiple-output (MIMO) systems, with m inputs $x_1(t), x_2(t), \ldots, x_m(t)$ and n outputs $y_1(t), y_2(t), \ldots, y_n(t)$. For MIMO systems, the PTF between the *i*th input and *k*th output is defined by the ratio of the $\mathcal{L}$-transform $Y_i(s)$ to the $\mathcal{L}$-transform $X_k(s)$, the other inputs being identically zero.

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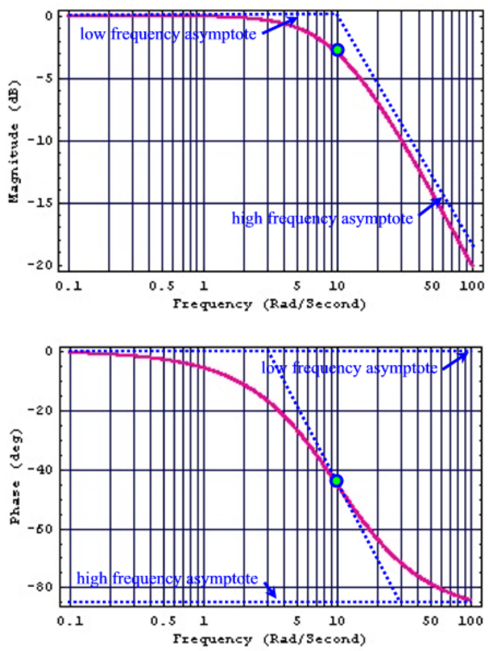

Figure 1: Bode diagrams of a TF with a real pole

1.5 State-space System

The state-space of a continuous-time system consists in two matrix equations: the state equation and the observation equation

$$\begin{aligned}\mathbf{x}'(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),\end{aligned}$$

where $\mathbf{u}(t)$ is a vector of inputs and $\mathbf{y}(t)$ a vector of outputs. The constant coefficients matrix $\mathbf{A}$ denotes the state matrix, $\mathbf{B}$ is the input matrix, $\mathbf{C}$ the output matrix and $\mathbf{D}$ the direct transmission matrix. The coefficient matrices $\mathbf{B},\ \mathbf{C},\ \mathbf{D}$ may be time dependent.
The block-diagram is represented in Figure 2.
The time-domain response is given by

$$\mathbf{x}(t) = e^{(t-t_0)\mathbf{A}}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{(t-\tau)\mathbf{A}}\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau.$$

The solution $\mathbf{x}(t)$ is the sum of a zero-input response given by the first element, and the zero-state response given by the second element. [5]

[5] For a TF $H(s)$ whose n poles are all real distinct, the PFE may be written as

$$H(s) = \frac{Y(s)}{U(s)} = \sum_{j=1}^{n}\frac{K_j}{s+p_j} \equiv \sum_{j=1}^{n} X_j(s),$$

where the p's denote the poles and K's, adequate constant values. Taking the $\mathcal{L}^{-1}$-transform , we deduce the state-space system

$$\begin{aligned}\mathbf{x}'(t) &= \begin{pmatrix} -p_1 & 0 & \dots & 0\\ 0 & -p_2 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & -p_n\end{pmatrix}\mathbf{x}(t) + \begin{pmatrix} K_1\\ K_2\\ \dots\\ K_n\end{pmatrix} u(t),\\ y(t) &= \begin{pmatrix} 1 & 1 & \dots & 1\end{pmatrix}\mathbf{x}(t).\end{aligned}$$

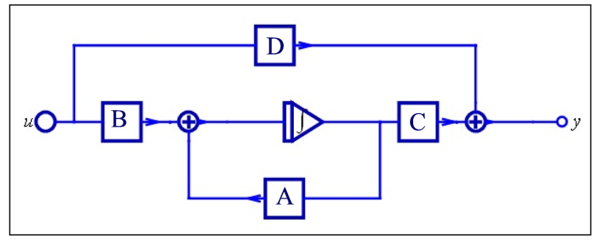

Figure 2: State-space block-diagram of a continuous-time system

1.6 System Stability: Concepts and Criteria

A system is stable if its impulse response tends towards zero, as time tends to infinity. A bounded signal is such that its absolute value is never greater than some existing quantity. A system is stable if every bounded perturbation on a system has a bounded response impact.

BIBO stability. According to the BIBO (Bounded Input- Bounded Output) stability, every bounded input $u(t)$ to a system results in a bounded output $y(t)$ over the time range $[t_0, \infty)$, for all initial conditions in t_0 and inputs. Then we have

$$\|u(t)\| \leq 1,\ t \geq t_0 \Rightarrow \|y(t)\| \leq k,$$

where k denotes some positive constant. For continuous-time system, the BIBO condition is the convergent integral

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty,$$

whereas, the discrete-time condition is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty.$$

Let a state-space SISO system be

$$\begin{aligned}x'(t) &= x(t) + u(t),\\ y(t) &= x(t).\end{aligned}$$

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The solution is

$$y(t) = x(0)e^t + e^t \int_0^t u(\tau)e^{-\tau} d\tau.$$

For the initial condition $x(0) = -1$ and a unit step function $u(t) = 1,\ t \in [0, \infty)$, y(t) is bounded with $y(t) = -1$ for all the time. However, if $u(t) = 0$ then $y(t)$ tends to infinity. The system is then not BIBO stable.

Lyapunov stability. According to the internal Lyapunov stability, the states of a system will remain bounded all the time, for any finite initial conditions. Let the state equation of a system be

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x}(t).$$

The solution of the system is

$$\mathbf{x}(t) = \Big(x_1(0)e^{2t}, x_2(0)e^{-t} \Big)$$

For $\mathbf{x}(0) = (0, 2)$, we obtain a bounded solution $\mathbf{x}(t) = (0, 2e^{-t})$ for all t. However, we get $\mathbf{x}(t) = (e^{2t}, 2e^{-t})$ for other initial conditions $\mathbf{x}(t) = (1, 0)$. The system is then Lyapunov unstable [6][7].

Root locus criterion. Let the general gain of a discrete-time system be

$$H(z) = \frac{C(z)}{R(z)}.$$

The zeros and poles of the system simultaneously satisfy the two equations $C(z) = 0$ and $R(z) = 0$. The TF may take the form

$$H(z) = H_0 \frac{\prod_{i=1}^{Z}(z - z_i)}{\prod_{i=1}^{P}(z - p_i)^{m_i}},$$

where Z is number of zeros, P the number of poles with multiplicity m_i. In linear form, we may write

$$N(z) + K.M(z) = 0,\ K \in (-\infty, \infty).$$

Nyquist stability criterion. The Nyquist stability criterion is based on the contour of the frequency response function. It combines the two types of Bode plots, the magnitude and the phase plots. The stability character of a system is deduced without computing the poles of the TF. The stability of a closed-loop system is deduced from the open-loop system's Nyquist plot. The negative closed-loop system will be stable if the contour does not encircle the point $(-1 + j0)$ in the s-plane. Let $F(s)$ be the open-loop TF, according to the Cauchy principle, the number of clockwise encirclements of the origin must be the difference between the number of zeros and the number of poles in the RHP. If a given Nyquist contour Γ_s encircles Z zeros and P poles, the resultant contour $\Gamma_{F(s)}$ encircles (clockwise) the point $(-1 + j0)$ $Z - P$ times. For a sysstem to be stable, we must have $Z = 0$ (i.e. no closed loop poles in the RHP). Hence, the number of clockwise encirclements about $(-1 + j0)$ must be equal to P.

2 Circuit Analysis of Environmental Pollution Systems

2.1 Green Solow-Swan Model

The green Solow-Swan model extends the reference one-sector model of economic growth [6, 12], by incorporating elements of environmental economics: the flow of pollution emissions and pollution concentration in stocks, the technological progress in pollution abatement. The model generates an EKC (Environmental Kuznets Curve) relationship between the pollution emissions and income per capita, and the pollution concentration and income per capita [3, 5]. The model consists of two blocks, one is the Solow-Swan growth model with environmental aspects, the other one is for environmental economics. The augmented model with exogenous technological progress in production and pollution abatement leads to a permanent growth with improved environmental quality.

A two blocks system. In the economic block of the model, a single good is used either for consumption or investment and we have a perfect competition. The savings rate is fixed, goods are produced according to a constant returns to scale strictly concave production function. The technological progress is labor

[6] Since the eigenvalues $s_1 = 2,\ s_2 = -1$ of the state matrix have a real part in right-half s-plane, the system is Lyapunov unstable.

[7] BIBO stability and Lyapunov stability are related, since the system's poles are a subset of its eigenvalues: for a continuous-time system to be stable, its poles must be in the left-half plane (LHP), as well as its eigenvalues. Let a numerical state-space system be

$$\begin{aligned} \mathbf{x}'(t) &= \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \\ y(t) &= \begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x}(t). \end{aligned}$$

Since the corresponding TF is $H(s) = \frac{1}{s+2}$, the unique real pole is negative and the system is BIBO stable. However, the same system is Lyapunov unstable, since the eigenvalues of the state matrix are $s_1 = 1,\ s_2 = -2$, with one value in the RHP.

augmenting. Capital stock accumulates by means of savings and depreciates at a given rate. The population growth is constant [8]. We obtain the first compact equation in intensive units for this block [9]

$$k'(t) = (1-\theta)sk(t)^{\alpha}(\delta+g+n)k(t), \quad (2)$$

where k denotes the capital stock per capita, θ abatement effort, s the fixed savings rate, δ the depreciation rate, g the exogenous labor-augmenting technological progress in goods, and n the constant growth rate of population. In the second block of the model, pollution is a joint product of output [5]. The emitted pollution is equal to the produced pollution less the abated pollution. Every unit of production activity generates several units of pollution and abatement is assumed to be a constant returns to scale activity. The stock of pollution varies with the emissions less a natural regeneration of the pollution stock. We obtain the second compact equation in intensive units for this block [10]

$$x'(t) = a(\theta)\Omega(t)k(t)^{\alpha} - (\eta+g+n)x(t),\ \eta > 0, \quad (3)$$

where $a(.)$ denotes the impact of abatement on pollution reduction, η the speed of natural regeneration. According to the function $a(\theta) = (1-\theta)^{\varepsilon}$, $\varepsilon > 1$, abatement is supposed to have a positive ($a'(\theta) > 0$) but diminishing marginal impact ($a''(\theta) < 0$) on pollution reduction. We also assume an exogenous technological progress in abatement, whose effect is to diminish the pollution $\Omega(t)$ at rate $g_A > 0$. Since, we have $\Omega'(t)/\Omega(t) = -g_A$, we deduce $\Omega(t) = \Omega(0)e^{-g_A t}$. The green Solow-Swan model is the system of non-linear FOCs (skipping the time arguments of k and x)

$$k' = (1-\theta)sk^{\alpha} - (\delta+g+n)k, \quad (4)$$

$$x' = (1-\theta)^{\varepsilon}\left(\Omega(0)e^{-g_A t}\right)k^{\alpha} - (\eta+g+n)x. \quad (5)$$

System dynamics. The solution of the system (4-5) is expressed via Hypergeometric functions. The Bernouilli non-linear ODE equation (4) is solved firstly, by using a substitution of function [11]. Taking $z(t) = k(t)^{1-\alpha}$ with $z'(t) = (1-\alpha)k(t)^{-\alpha}k'(t)$, we obtain the linear ODE

$$z'(t)+(1-\alpha)(\delta+g+n)z(t) = (1-\alpha)(1-\theta)s. \quad (6)$$

Given the initial condition $k(0) = k_0$, the solution of equation (6) is

$$k(t) = \left(\left(k_0^{1-\alpha} - \frac{(1-\theta)s}{\delta+g+n}\right)e^{-(1-\alpha)(\delta+g+n)t} + \frac{(1-\theta)s}{(\delta+g+n)}\right)^{\frac{1}{1-\alpha}}.$$

Inserting the solution for $k(t)$ into equation (5), we get a linear ODE of the form [12]

$$x'(t) + (\delta+g+n)x(t) = \varphi(t).$$

[8] Ferrara and Guerrini (2009) extend the Solow-Swan model with a logistic population growth, in Recent Advances in Mathematics and Computers in Business and Economics, WSEAS, Prague, March 23-25, pp. 17-20.

[9] Assuming a Cobb-Douglas production function

$$Y(t) = \big(B(t)L(t)\big)^{1-\alpha}K(t)^{\alpha},\ \alpha \in (0,1),$$

where Y is the aggregate output, L the labor, K the physical capital stock. To be expressed in intensive units, all the variables are divided by $\big(B(t)L(t)\big)$. The production function then takes the simplified form $y(t) = k(t)^{\alpha}$. The same kind of transformation is applied to the basic accumulation equation

$$K'(t) = (1-\theta)sY(t) - \delta K(t),$$

where θ denotes the proportion of production activity dedicated to abatement. Equation (2) is then obtained after simple algebraic manipulations.

[10] The stock of pollution is related to the flow of emissions and to the natural regeneration by

$$X'(t) = E(t) - \eta X(t),$$

where E denotes the flow of emissions and X the stock of pollution. The emitted pollution at time t is

$$E = -\Omega A(Y, Y^A) = \Omega Y\left(1 - A\big(1, \frac{Y^A}{Y}\big)\right) = \Omega Y a(\theta),$$

where a level A of abatement will remove ΩA units of pollution. To be expressed in intensive units, all the variables are divided by $(BL$, s.a. with $x = X/(BL)$. Thereafter, equation (3) is obtained by elementary algebraic manipulations.

[11] The general form of the Bernouilli ODE is

$$y' + P(x)y = Q(x)y^n,\ n \neq 0 \text{ and } n \neq 1.$$

Substituting $z = y^{1-n}$ implies $y = z^{\frac{1}{1-n}}$ with $y' = \frac{1}{1-n}z^{\frac{n}{1-n}}z'$ and dividing both sides of the equation by $z^{\frac{n}{1-n}} \neq 0$, will change the original to

$$\frac{1}{1-n}z' + P(x)z = Q(x).$$

[12] To solve an ODE of the form

$$y' + Py = Q(x),$$

where P is a constant, we use the integrating factor (Giordano & Weir (1991)[7]). The procedure is: first calculate the integrating factor $\mu(x) = e^{\int P dx} = e^{Px}$, then multiply the right side of the original by $\mu(x)$ and integrate $\int \mu(x)Q(x)dx + C$ and write the general solution $\mu(x)y(x) = \int \mu(x)Q(x)dx + C$ or

$$y(x) = e^{-Px}\int e^{Px}Q(x)dx + Ce^{-Px}.$$

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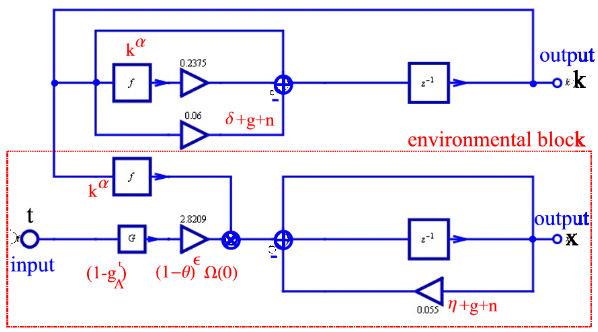

Figure 3: Block-diagram of the green Solow-Swan model

The heavy form for $\varphi(t)$ introduces hypergeometric functions in the solution. Hypergeometric function ${}_2F_1$ is represented by the series expansion [13] [2]

$$ {}_2F_1(z;a,b,c) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}, $$

where the Euler Gamma function satisfies

$$ \Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt. $$

The block-diagram of the model is shown in Figure 3 with time as input and two outputs, the stock of pollution and the capital stock. The corresponding discrete-time system is

$$ \begin{aligned} k_{t+1} &= k_t + (1-\theta)\sigma k_t^{\alpha} - (\delta+g+n)k_t, \\ x_{t+1} &= x_t + (1-\theta)^{\varepsilon}(1-g_A)^t k_t^{\alpha} - (\eta+g+n)x_t. \end{aligned} $$

The simulation with a ramp function as input shows a typical humped curve (Figure 4).

3 Circuit Analysis of Fishery Environmental Systems

3.1 Fisheries Dynamics

Open-access fishery model. The open-access fishery model [10] consists in two dynamic equations:

[13] The special hypergeometric function ${}_2F_1$ (see Nikiforov & Uvarov (1988)[9]) is a solution of the differential equation

$$ z(1-z)y'' + (c-(a+b+1)z)y' - aby = 0. $$

It is also the integral

$$ \frac{\Gamma_{(c)}}{\Gamma_{(b)}\Gamma_{(c-b)}} \times \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt. $$

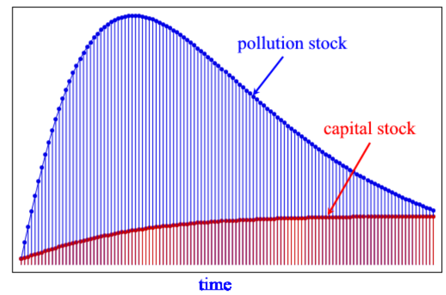

Figure 4: Simulation of the discrete-time green Solow-Swan model

one equation represents the growth rate of the fish resource, the other equation describes the responsiveness of the industry size to the profitability of the industry. The open-access assumption means the acceptance of the main characteristics of the perfect competition model: a large number of fishing firms and no barrier to the entry into and to exit from the fishery industry [10]. As a main consequence, the fishing firms cannot influence the market price, taken as given (exogenous). In compact form, the two-dimensional nonlinear system of ODEs is

$$ S'(t) = g\left(1 - \frac{S(t)}{S_{max}}\right)S(t) - eQ(t)S(t), \quad (7) $$

$$ Q'(t) = \delta\Big(ePQ(t)S(t) - wQ(t)\Big), \quad (8) $$

where the variables are the stock of fish S and the effort Q. The parameters are e the catch fishing coefficient, $g > 0$ the potential biomass growth, P the market price, S_{max} the resource carrying capacity, w the unit cost of harvesting effort and $\delta > 0$ the responsiveness intensity of industry size to the profitability. Equation (7) describes the net growth of the fish stock, which is a biological logistic (density-dependent) function [14] less a simple harvesting function [15]. Equation (8) describes the fishing effort dynamics [16]

[14] The alternative Gompertz growth function for biological evolution is

$$ f(S) = g\left(\ln \frac{S_{max}}{S(t)}\right) S(t). $$

[15] According to the harvesting function, the harvested quantity H per unit effort Q is a multiple e of the stock size S. Then we get the second member of equation (7), from the equivalence $\frac{H}{Q} = eS \Leftrightarrow H = eQS$.

[16] To find equation (8), we start from the assumption that efforts will be incited by the profitability of the industry, according to $Q'(t) = \delta NB$, where NB denotes the net profit: effort will continue as long as fishing is profitable, and will cease when profits tend to be negative. The net profit is the difference between the

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Steady state equilibrium. The steady-state equilibrium points are obtained by solving simultaneously the system $S'(t) = 0$ and $Q'(t) = 0$. We get the three equilibrium points: $(0,0)$, $(S_{max}, 0)$ and

$$\left(\bar{S} = \frac{w}{eP},\ \bar{Q} = \frac{g(ePS_{max} - w)}{e^2PS_{max}}\right).$$

Linearizing equations (7-8) and evaluating the equations at the third steady-state equilibrium point, we have the matrix equation for the system

$$\begin{pmatrix} S'(t) \\ Q'(t) \end{pmatrix} = \begin{pmatrix} -\frac{gw}{ePS_{max}} & -\frac{w}{P} \\ g\delta(P - \frac{w}{eS_{max}}) & 0 \end{pmatrix} \begin{pmatrix} S(t) - \bar{S} \\ Q(t) - \bar{Q} \end{pmatrix}$$

Trajectories are converging in the neighborhood of the equilibrium point $(.5, 8.3)$ for which the parameter values are taken from [10]

$$e = .009,\ g = .15,\ P = 200,\ S_{max} = 1,\ w = .9,\ \delta = .4$$

The numerical matrix of the system is

$$A = \begin{pmatrix} -.075 & -.0045 \\ 6 & 0 \end{pmatrix}.$$

Since the roots of the characteristic equation $|A - sI| = 0$, valued at this equilibrium point, are complex with a negative real part $s = -.0375 \pm .16j$, the equilibrium is proved to be a locally asymptotically stable focus.

Block-diagram approach. A block-diagram (Figure 5) is deduced from the discrete version of the nonlinear model (equations (7-8)). We have the system

$$\begin{aligned} S_{k+1} &= S_k + g\left(1 - \frac{S_k}{S_{max}}\right)S_k - eQ_kS_k, \\ Q_{k+1} &= Q_k + \delta(PeS_k - w)Q_k. \end{aligned}$$

Three types of input signals are tested in the simulations of Figure 6: a 20 periods delayed impulse signal, a unit step signal and a unit white noise which superposed to the initial market price level at 200.

3.2 Fisheries Control

The maximization problem. The maximization problem of the fishery industry [11] is to maximize the total discounted profits over the time period $[0, T]$ subject to the biological growth of the fish resource.

total revenue from the sales $PH(t)$and a linear total cost $wQ(t)$. Replacing these elements into the equation for $Q'(t)$ yields equation (8).

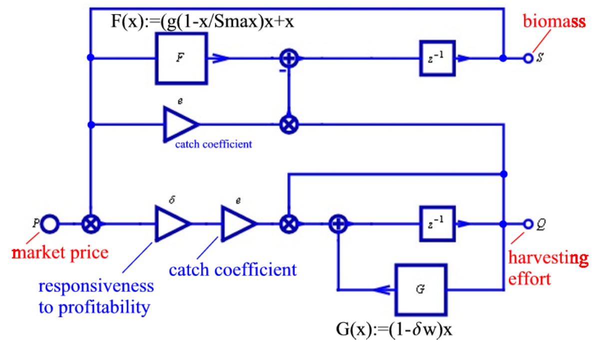

Figure 5: Block-diagram of the fishery open-access model

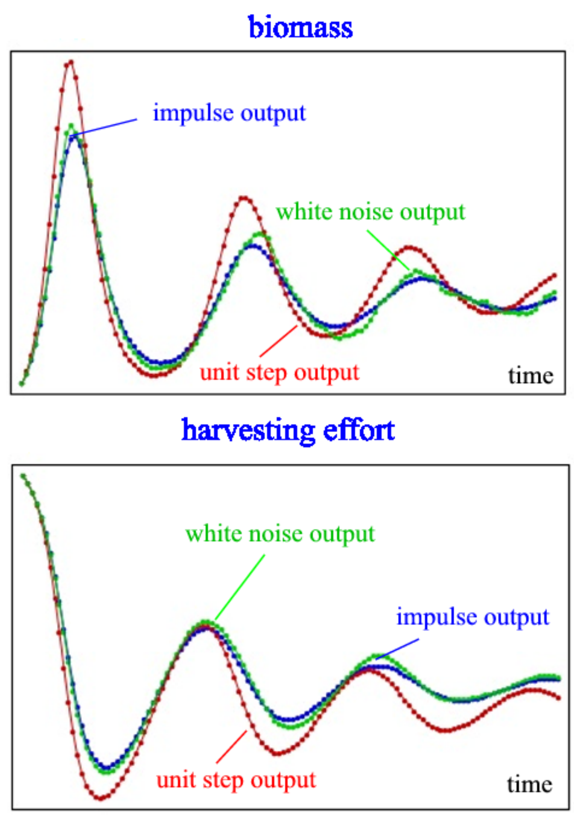

Figure 6: Simulation output of the fishery open-access model

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The discounted profit at time t may be written (skipping the time argument) $e^{-\rho t}V(S,Q,t)$ where $e^{-\rho t}$ denotes the discount factor, ρ the discount rate, Q the harvesting effort and S the stock of fish at time t. Let the harvesting function be $h(S,Q)$, a function of stock and effort. The profits $V\big(S,Q,t\big)$ are simply the total revenue from selling the fish $P.h(S,Q)$ less the total cost wQ. For this problem, the harvesting effort Q will be the control variables, and S the state variable. The optimal control problem of the fishery industry is

$$\text{maximize}_{Q(t)}$$
$$\int_0^T e^{-\rho t}\Big(P.h\big(S(t),Q(t)\big)-wQ(t)\Big)dt+F(S^T)$$
$$\text{subject to}$$
$$S'(t)=f\big(S(t)\big)-h\big(S(t),Q(t)\big),$$
$$S(0)=0,\ S(T)=S^T.$$

The Hamiltonian for this problem is

$$\mathcal{H}\big(S(t),Q(t),\pi(t),t\big)=$$
$$e^{-\rho t}\Big(P.h\big(S(t),Q(t)\big)-wQ(t)\Big)+$$
$$\pi(t)\Big(f\big(S(t)\big)-h\big(S(t),Q(t)\big)\Big),$$

where the π's denote the costate variables, which are the shadow price or price of the fish resource. The necessary FOCs are (skipping the time arguments)

$$\frac{\partial\mathcal{H}}{\partial Q}=e^{-\rho t}\Big(P-\frac{\partial h}{\partial Q}-w\Big)-\pi\frac{\partial h}{\partial Q}=0,$$
$$\pi'(t)=-\frac{\partial\mathcal{H}}{\partial S}\ =-\Big(e^{-\rho t}P\frac{\partial\mathcal{H}}{\partial S}+\pi f'(S)-\pi\frac{\partial\mathcal{H}}{\partial S}\Big),$$
$$S'(t)=\frac{\partial\mathcal{H}}{\partial\pi}=f(S)-h\big(S,Q\big),$$
$$S(0)=0,\ \pi(T)=\frac{\partial F}{\partial S^T}.$$

Let define $\mu(t)=e^{\rho t}\pi(t)$, we get $\pi'(t)=e^{-\rho t}\mu'(t)-\rho e^{-\rho t}\mu(t)$. The necessary FOCs are then transformed to

$$(P-\mu)\frac{\partial\mathcal{H}}{\partial Q}=w, \quad (9)$$
$$\mu'=\big(\rho-f'(S)\big)\mu-(P-\mu)\frac{\partial\mathcal{H}}{\partial S}, \quad (10)$$
$$S'=f(S)-h\big(S,Q)\big), \quad (11)$$

Equation (9) is the maximum principle condition, equation (10) the portfolio balance condition whose interpretation states [11] a comparison between a net advantage obtained from selling the fish (the first term) and the net revenue of holding the fish (second term). Equation (11) is the dynamic constraint of the control problem. The maximum principle indicates that the current value Hamiltonian is maximized if the marginal net revenue from harvesting efforts equals the marginal cost for such efforts (see Shone [11]: 658-661, for further discussion). Suppose that the functional for the harvesting function and the biological form take the forms (skipping the time arguments)

$$h(Q)=eQ^{\varepsilon},\ f(S)=g\big(1-\frac{S}{S_{max}}\big)S.$$

The necessary FOCs are

$$e\varepsilon(P-\mu)Q^{\varepsilon-1}=w, \quad (12)$$
$$\mu'(t)=\mu\Big(\rho-\big(g-\frac{2gS}{S_{max}}\big)\Big), \quad (13)$$
$$S'(t)=g\Big(1-\frac{S}{S_{max}}\Big)S-eQ^{\varepsilon}. \quad (14)$$

Phase-diagram analysis in the state-costate plane (S,μ). Eliminating Q from the three FOCs (equations (12-14)) by using the condition (12), we obtain the pair of differential equations

$$S'(t)=g\big(1-\frac{S(t)}{S_{max}}\big)S(t)-e\Big(\frac{e\varepsilon\big(P-\mu(t)\big)}{w}\Big)^{\frac{\varepsilon}{1-\varepsilon}}, \quad (15)$$
$$\mu'(t)=\Big(\rho-g+2g\frac{S(t)}{S_{max}}\Big)\mu(t). \quad (16)$$

The steady-state equilibrium point is obtained by solving simultaneously $S'(t)=0$ and $\mu'(t)=0$. We get

$$\bar{S}=\frac{(g-\rho)S_{max}}{2g},\ \bar{\mu}=P-\frac{w\ 4^{\frac{\varepsilon-1}{\varepsilon}}\times\Big(\frac{S_{max}(g^2-\rho^2)}{e\ g}\Big)^{\frac{1-\varepsilon}{\varepsilon}}}{e\ \varepsilon}.$$

The characterization of the steady-state point is studied locally by linearizing equations (15-16) at this point. We have the matrix equation

$$\begin{pmatrix} S'(t)\\ \mu'(t)\end{pmatrix}=\mathbf{A}\begin{pmatrix} S(t)-\bar{S}\\ \mu(t)-\bar{\mu}\end{pmatrix},$$

where

$$\mathbf{A}=\begin{pmatrix} g\big(1-\frac{2S(t)}{S_{max}}\big) & \frac{e\ \varepsilon\times\big(\frac{e\ \varepsilon(P-\mu(t))}{w}\big)^{\frac{\varepsilon}{1-\varepsilon}}}{P-P\ \varepsilon-\mu+\varepsilon\ \mu(t)}\\ \frac{2g\ \mu(t)}{S_{max}} & g\big(\frac{2S(t)}{S_{max}}-1\big)+\rho\end{pmatrix}.$$

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Taking the parameter values [11]

$$e = 5,\ g = .2,\ P = 10,\ S_{max} = 100,\ w = 8,\ \varepsilon = .5,\ \rho = .1$$

the valued matrix at equilibrium is

$$\begin{pmatrix} .1 & 1.5625 \\ .0304 & 0 \end{pmatrix}.$$

The characteristics of the steady-state equilibrium are given by the eigenvalues and eigenvectors of the valued matrix. Since the eigenvalues are real roots of opposite signs with $s_1 = .273$ and $s_2 = -.1736$, the equilibrium point is a saddle point, depicted in Figure 7. The eigenvectors $(.9939, .1104)$ and $(-.9850, .1725)$ allow for the determination of the saddle paths, either a stable or an unstable arm.

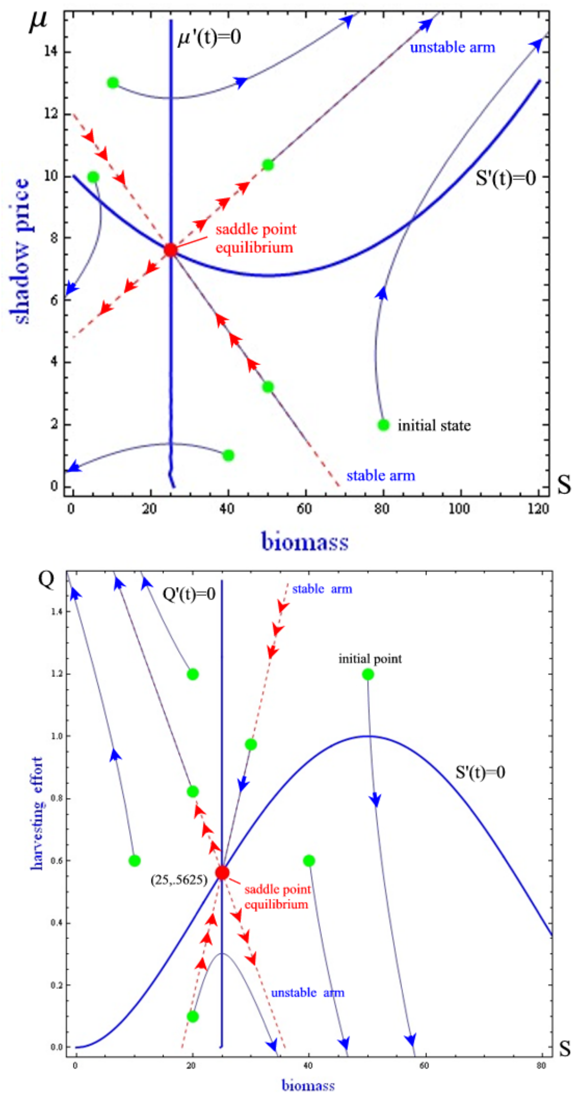

Figure 7: Trajectories of the control fishery model in the (S,μ)-plane and (S,Q)-plane

Phase-diagram analysis in the state-control plane (S,Q). A similar approach shows a saddle point equilibrium (see Figure 7).

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