Multilevel Multiagent Nonconvex Optimization

An Introduction to the Nash-Stackelberg Equilibrium Solutions

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***Abstract*—This article introduces to the hierarchical nonconvex optimization techniques with multiple agents at multiple levels. In practice, many areas in engineering, economic policies, transportations and regional planning have already been concerned with such a decision-making process. The multilevel programming was defined in the mid-1970s. A particular emphasis is made on the equilibrium determination in the Nash-Stackelberg static game, with one leader at the upper-level and multiple followers at the lower-level. Examples from the literature have been revisited by using the Wolfram/*Mathematica* software V.8.0.0.-10.**

Keywords: bilevel programming problem; decision-making; multilevel Optimization; Nash-Stackelberg equilibrium

I. Introduction to Bilevel Programming

A. Decision Making in Hierarchical Organizations

Suppose an organization with two levels of hierarchy in decision making : the leader at the upper-level and the follower at the lower-level (Bard [5], Cruz [13]). This may be the case of a private company, in which the top management has overall economic objectives and where specialized divisions have productivity and marketing objectives (for engineering and economical applications, see Bard [5], Dempe [15, 16]). The decision process is such that the two levels proceed sequentially, in the sense that the leader may influence the decision of the follower. The follower observes the leader's decision and reacts optimally.

This problem is similar to the static noncooperative two-person game by Stackelberg (Pfähler [24], von Stackelberg [28]). The two players optimize their own payoff function by controlling their decision variables. Both players have a perfect information about the objectives and strategies of the opponent. The leader plays first, but must anticipate all the possible reactions of his opponent and the follower reacts optimally.

B. Decision-Making Problem

The formulation of this decision-making problem refers to a bilevel programming (BLP) problem, in which the constraint region is implicitly determined by another optimization problem. Let the decision variables controlled by the leader be $\mathbf{x} \in X \subseteq \mathbb{R}^n$ and the follower's decision variables be $\mathbf{y} \in Y \subseteq \mathbb{R}^m$. A general form of the BLP may be written with no upper-level constraints, such as

$$\left[\begin{array}{l} \underset{\mathbf{x}\in X}{\text{minimize}}\ \ F(\mathbf{x},\mathbf{y}) \equiv \mathbf{c}_1^T\mathbf{x} + \mathbf{d}_1^T\mathbf{y} \\ \quad s.t. \left[\begin{array}{l} \underset{\mathbf{y}\in Y}{\text{minimize}}\ \ f(\mathbf{x},\mathbf{y}) \equiv \mathbf{c}_2^T\mathbf{x} + \mathbf{d}_2^T\mathbf{y} \\ s.t.\ \ \mathbf{g}(\mathbf{x},\mathbf{y}) \equiv \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} - \mathbf{b} \le 0 \end{array}\right. \end{array}\right.$$

where the outer and inner objective functions are $F, f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ and the inner constraints $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^q$. All the functions are assumed to be continuous and C^2. Moreover, the sets X and Y will place additional restrictions on the variables such as bounds, nonnegativity, or integrality. Under convexity and regularity conditions, using the Karush-Kuhn-Tucker (KKT) conditions for the second level problem, the BLP is reformulated as a single nonlinear optimization problem

$$\left.\begin{array}{r} \underset{\mathbf{x},\mathbf{y},\mathbf{u}}{\text{minimize}}\ \ \mathbf{c}_1^T\ \mathbf{x} + \mathbf{d}_1^T\mathbf{y} \\ s.t.\ \nabla_{\mathbf{y}}\mathcal{L}(\mathbf{x},\mathbf{y},\mathbf{u}) = \mathbf{d}_2^T + \mathbf{u}^T\mathbf{B} = 0 \\ \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} - \mathbf{b} \le \mathbf{0} \\ u_i\left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} - \mathbf{b}\right)_i = 0,\ i = 1,\ldots,q \\ \mathbf{x} \ge \mathbf{0},\ \mathbf{y} \ge \mathbf{0},\ u_i \ge 0,\ i = 1,\ldots,q \end{array}\right]$$

where $\mathcal{L}\left(\mathbf{x},\mathbf{y},\mathbf{u}\right) = f(\mathbf{x},\mathbf{y}) + \sum_{i=1}^{q} u_i g_i(\mathbf{x},\mathbf{y})$ denotes the Lagrangean associated with the lower-level problem and $\mathbf{u}$ the multipliers. Different algorithms have been proposed to replace the complementarity slackness and Lagrangean constraints, by adding new variables and constraints, or using the branch and bound (B&B) algorithm.

C. Introductory Example

Example 1. A linear BLP problem with two players is taken from Bard [5], to define the BLP constraint region S, the follower's feasible set $S(x)$, the rational reaction set $RR(x)$, and the projection $S(X)$ of S onto the leader's decision space. The controlled variables by players are respectively $x \in X \subseteq \mathbb{R}_{\geq}$ and $y \in Y \subseteq \mathbb{R}_{\geq}$. The numerical BLP example is shown in Fig.1. The BLP problem is

$$\begin{bmatrix} \min\limits_{x\geq 0}\text{imize} \;\; F(x,y) \equiv x - 4y \\ s.t. \begin{bmatrix} \min\limits_{y\geq 0}\text{imize} \;\; f(y) \equiv y \\ s.t. \;\; g_1(x,y) \equiv -x - y + 3 \leq 0 \\ g_2(x,y) \equiv -2x + y \leq 0 \\ g_3(x,y) \equiv 2x + y - 12 \leq 0 \\ g_4(x,y) \equiv -3x + 2y + 4 \leq 0 \end{bmatrix} \end{bmatrix}$$

The polyhedron S is the set

$$S = \left\{ (x,y) : x \in X, y \in Y, g_i(x,y) \leq 0, i \in \mathbb{N}_4 \right\}.$$

For fixed $x \in X$, the follower's feasible region is the set $S(x) = \left\{ y \in Y : y \geq 3 - x, y \leq 2x, y \geq -12 + 2x, y \leq 2 - 1.5x \right\}$ The follower's reaction set is defined by the set $RR(x) = \left\{ y \in \arg\min f(x,\hat{y}) : \hat{y} \in S(x) \right\}$ .The piecewise inductible region $IR = \left\{ (x,y) : x \geq 0, y \in RR(x) \right\}$ is the feasible set of the leader. The projection of S onto the leader's decision space is the set $S(X) = \left\{ x \in X : \exists y \in Y, \mathbf{g}(x,y) \leq 0 \right\}$. We deduce $S(X) = \left\{ x \in [1,4] \right\}$.

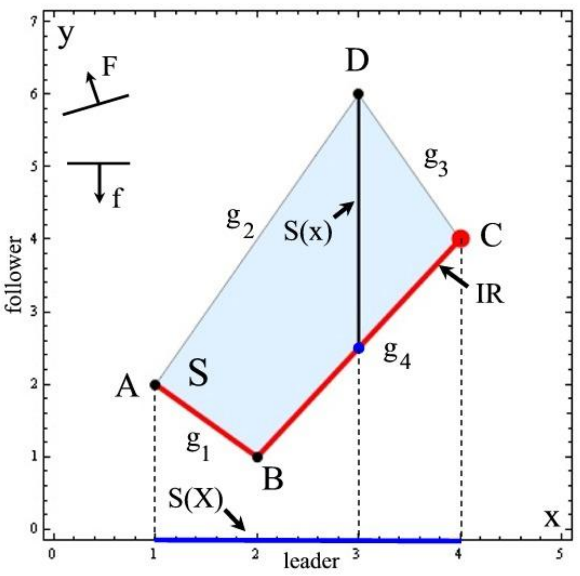

Figure 1 Feasible region and global optimum

The polyhedron S shows the constraint region.

Theorem 1. The inductible region IR is equivalently a piecewise linear equality constraint comprised of supporting hyperplanes of S. **Proof.** see Bard [5], p. 199.■

Theorem 2. The solution $\left(\mathbf{x}^*, \mathbf{y}^* \right)$ of a linear BLP occurs at a vertex of S. **Proof.** see Bard [5], p. 200. ■

The payoff matrix in TABLE 1 is used to search for the optimum. For feasible points, the payoffs are (f, F) and ∞ for non feasible points. For extreme points on the IR, the payoffs are highlighted.

TABLE 1 PAYOFF MATRIX OF EXAMPLE 1

		Leader (x)				min
		1	2	4	6	
Follower (y)	1	∞	**(2,-7)**	∞	∞	2
	2	**(1,-2)**	(2,-6)	(4,-14)	∞	**1^***
	3	∞	∞	(4,-13)	(6,-21)	4
	4	∞	∞	**(4,-12)**	∞	4^+
min		-2	-7	-14^+	**-21^***	

The optima occur at extreme points of inductible region. The global optimal solution is achieved at C (4,4) with $\left(f^*, F^* \right) = (4, -12)$ There is no Pareto optimal solution. A local optimum is at A(1,2).

II. MULTIAGENT - MULTILEVEL PROGRAMMING

A. Multilevel Multiagent Decision Making System

The structure of an N-agent L-level programming problem is shown in Fig. 2. This presentation is inspired from Yang and Bialas [31]. The hierarchical decision system consists of L levels. At each level $k = 1, \ldots, L$, the number of agents (or divisions) is n_k. Agent i at level k is denoted by $\mathcal{D}_i^k \in \mathcal{L}^k$ and all other agents at the same level will be $\mathcal{D}_{-i}^k$. Agent $\mathcal{D}_i^k$ is controlling $\mathbf{x}_i^k \in X_i^k \subseteq \mathbb{R}^{m_{k_i}}$ with m_{k_i} decision variables. The decision variables for level k are $\mathbf{x}^k = \left(\mathbf{x}_1^k, \ldots, \mathbf{x}_{n_k}^k \right) \in X^k = \bigtimes_{i=1}^{n_k} X_i^k$. The total number of decision variables in the system is $N = \sum_{k=1}^{L} \sum_{i=1}^{n_k} m_{k_i}$ .The system is a nested collection of Nash equilibrium problems: within each level, agents play an $n-$ person nonzero-sum game and between levels, the decision process is similar to an $n-$ person Stackelberg game (Yang and Bialas [31]). Every agent has a perfect information. They are perfectly informed about the decisions at upper levels, but not at their levels or below. The agents of one level will also influence the other agents at lower levels, via their objective functions and the sets of feasible decisions.

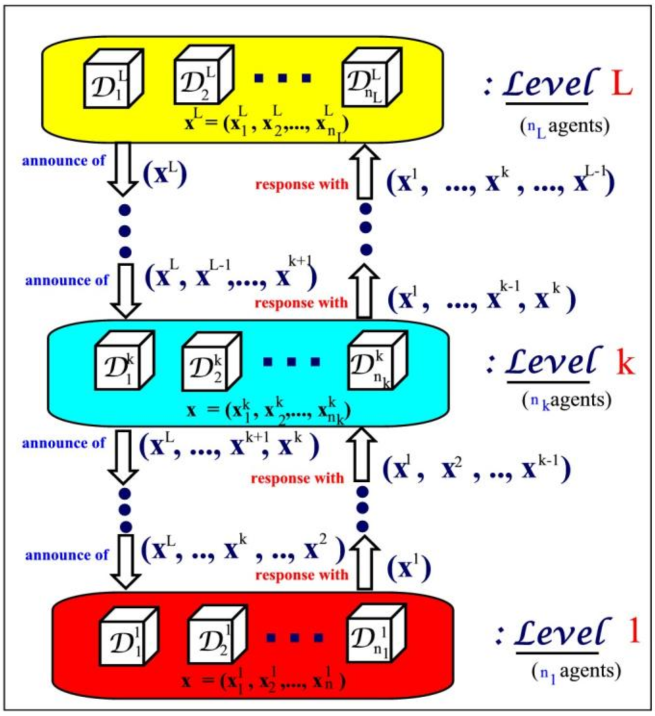

Figure 2 Multilevel decision making system

B. *Nash-Stackelberg Equilibrium Solution*

Let the objective functions of agent $\mathcal{D}_i^k$ be $f_i^k : \mathbb{R}^N \mapsto \mathbb{R}$ for $i = 1, ..., n_k$ and $k = 1, ..., L$.

Definition 1 (Nash equilibrium responses). *The Nash equilibrium responses at level* k *of* $f_1^k, ..., f_{n_k}^k$ *over the compact set* S *, for each* $k = 1, ..., L-1$*, is defined as.*

$$\Psi^k(S) \equiv \left\{ \hat{w} \in S \;\middle|\; \begin{array}{l} f_1^k(\hat{w}) = \underset{\mathbf{x}_1^k, \hat{\mathbf{x}}_{-1}^k : \hat{\mathbf{x}}^L \ldots \hat{\mathbf{x}}^{k+1}}{\text{max imize}} \; f_1^k(w) \\ \vdots \\ f_{n_k}^k(\hat{w}) = \underset{\mathbf{x}_{n_k}^k, \hat{\mathbf{x}}_{-n_k}^k : \hat{\mathbf{x}}^L \ldots \hat{\mathbf{x}}^{k+1}}{\text{max imize}} \; f_{n_k}^k(w) \end{array} \right\}.$$

Assumption 1. *The parametric problem for the* n_k *agents* $\mathcal{D}_i^k \in \mathcal{L}^k$ *has no multiple equilibrium solution. That is, for fixed values* $\left(\hat{\mathbf{x}}^L, \hat{\mathbf{x}}^{L-1}, \ldots, \hat{\mathbf{x}}^{k+1}\right)$*, the set* $\Psi^k(S)$ *has at most one element (Yang and Bialas [31]).*

Definition 2 (Equilibrium solution). *Let the level* k *feasible set be* $S^k \equiv \Psi^{k-1}\left(S^{k-1}\right)$ *for any given* $\left(\hat{\mathbf{x}}^L, \hat{\mathbf{x}}^{L-1}, \ldots, \hat{\mathbf{x}}^{k+1}\right)$*. An equilibrium solution* $\mathbf{x} \in S^k$ *is*

$$\mathbf{x} \equiv \left(\mathbf{x}^1, \ldots, \mathbf{x}^{k-1}, \mathbf{x}_i^k, \mathbf{x}_{-i}^k, \mathbf{x}^{k+1}, \ldots, \mathbf{x}^L\right)$$

Definition 3 (Stackelberg feasibility). If the equilibrium solution $\mathbf{x}$ also satisfies the rational responses of lower levels $\mathcal{L}^{k-1}, \ldots, \mathcal{L}^1$, then it is Stackelberg feasible.

The programming problem to be solved simultaneously by every $\mathcal{D}_i^k \in \mathcal{L}^k$ to get a Nash-Stackelberg solution is

$$\left.\begin{array}{ll} \underset{\mathbf{x}_i^k, \mathbf{x}_{-i}^k : \hat{\mathbf{x}}^L, \hat{\mathbf{x}}^{L-1}, \ldots, \hat{\mathbf{x}}^{k+1}}{\text{max imize}} & f_i^k(\mathbf{x}), \; i = 1, \ldots, n_k \\ & s.t. \; \mathbf{x} \in S^k \subset \mathbb{R}^N \end{array}\right]$$

The complete explicit form of the program is (Yang and Bialas [31])

$$\mathcal{P}^L \left[\begin{array}{l} \underset{\mathbf{x}_i^L, \mathbf{x}_{-i}^L}{\text{max imize}} \, f_i^L(\mathbf{x}), \; i = 1, \ldots, n_L \\ \quad where \; \mathbf{x}^{L-1}, \ldots, \mathbf{x}^1 \; solve: \\ \mathcal{P}^{L-1} \left[\begin{array}{l} \underset{\mathbf{x}_i^{L-1}, \mathbf{x}_{-i}^{L-1} : \mathbf{x}^L}{\text{max imize}} \, f_i^{L-1}(\mathbf{x}), \; i = 1, \ldots, n_{L-1} \\ \quad where \; \mathbf{x}^{L-2}, \ldots, \mathbf{x}^1 \; solve: \\ \quad \ddots \\ \mathcal{P}^2 \left[\begin{array}{l} \underset{\mathbf{x}_i^2, \mathbf{x}_{-i}^2 : \mathbf{x}^L, \ldots, \mathbf{x}^3}{\text{max imize}} \, f_i^2(\mathbf{x}), \; i = 1, \ldots, n_2 \\ \quad where \; \mathbf{x}^1 \; solves: \\ \mathcal{P}^1 \left[\begin{array}{l} \underset{\mathbf{x}_i^1, \mathbf{x}_{-i}^1 : \mathbf{x}^L, \ldots, \mathbf{x}^2}{\text{max imize}} \, f_i^1(\mathbf{x}), \; i = 1, \ldots, n_1 \\ \qquad s.t. \; \mathbf{x} \in S \end{array}\right. \end{array}\right. \end{array}\right. \end{array}\right.$$

C. *Finding Nash-Stackelberg Solutions*

Several algorithms have been proposed in the literature for solving linear and nonlinear BLPs. (Campelo and Scheimberg [10]). These algorithms are relevant to three main approaches: the extreme vertex exploration in the linear case, the reformulation of the original problem, the descent methods. Using heuristic algorithms such as simulated annealing, genetic algorithms (Deb [14], Liu [21], Wang et al. [29]) is a recent tendency. Some of the methods are oriented to local optimization techniques such as the penalty and barrier function method, others are global optimization techniques. Optimization principles are from Antoniou and Lu [3], Bazaraa et al. [6], Bertzekas [7], Gill et al. [20]. Bilevel optimization is from Bard [5], Colson et al. [12], Dempe [15, 16], Floudas [18].

- **Extreme-point approach**. According to **Theorem 2**. , some form of vertex enumeration may be employed for linear BLPs. The algorithms are based on the vertex enumeration and evaluation of extreme points of the constraint region. The *K*th best method (Bialas and Karwan [8], Candler and Townsley [11], Shi et al. [25, 26] considers bases of the relaxed problem (complementarity term omitted), sorted in increasing order of the upper level objective function values. .

- **Reformulation techniques**. Suppose that the lower-level problem is convex and regular. Then, the original problem is transformed into a single optimization problem as in section I.B, by employing the KKT conditions of the lower-level problem (Lu et al. [22, 23]. However, the nonconvexities, that occur in the linear complementarity slackness constraint, require some further transformations, such as: adding new variables and constraints and solving a mixed integer programming technique (Fortuny-Amat and McCarl [19]) or using a B&B enumeration technique (Bard and Falk [4]). Also, by using the KKT conditions for the lower-level problem, the parametric complementary pivot (PCP) has been proposed: in Bialas et al. [9] it is updating a parameter α, which bounds the upper-level objective function value. The lower-level problem may also be replaced by a penalized problem. Penalty methods in Aiyoshi and Shimizu [1,2], White and Amandalingam [30] introduce the duality gap of the follower's problem into the leader's problem.

$$
\mathcal{P}^2 \left[
\begin{array}{l}
\underset{x_1^2, x_{-1}^2}{\text{maximize}} \quad f_1^2(\mathbf{x}) \equiv 2x_1^1 + 0.7x_1^2 - 0.6x_2^2 \\
\underset{x_2^2, x_{-2}^2}{\text{maximize}} \quad f_2^2(\mathbf{x}) \equiv -2x_1^1 + 2x_1^2 - 1.5x_2^2 \\
\qquad\qquad\qquad \textit{where } x_1^1 \textit{ solves:} \\
\mathcal{P}^1 \left[
\begin{array}{rl}
\underset{x_1^1 : \hat{x}_1^2, \hat{x}_{-2}^2}{\text{maximize}} & f_1^1(\mathbf{x}) \equiv x_1^1 + 0.8x_1^2 + 1.2x_2^2 \\
s.t. & g_1(\mathbf{x}) \equiv -3 + x_1^1 + x_1^2 + x_2^2 \le 0 \\
 & g_2(\mathbf{x}) \equiv 1 + x_1^1 - x_1^2 - x_2^2 \le 0 \\
 & g_3(\mathbf{x}) \equiv -1 + x_1^1 + x_1^2 - x_2^2 \le 0 \\
 & g_4(\mathbf{x}) \equiv -1 + x_1^1 - x_1^2 + x_2^2 \le 0 \\
 & g_5(\mathbf{x}) \equiv -0.5 + x_1^1 \le 0 \\
 & x_1^1, x_1^2, x_2^2 \ge 0
\end{array}
\right.
\end{array}
\right.
$$

- **Descent methods**. Assume that the optimal solution at lower-level is unique, and define an implicit function $\mathbf{y}$ (the decision variables of the leader) of $\mathbf{x}$ (the decision variables of the follower). Given a feasible point, an attempt is made to find a feasible direction along which the upper-level objective decreases. The main issue is the availability of the gradient at the feasible point (Falk and Liu [17], Shi et al. [25, 26], Vicente et al.[27].).

III. NASH-STACKELBERG GAME

A. Two Level Decision System

Example 2. This simple application is taken from Yang and Bialas [31]. The hierarchical system consists of two levels, with one leader at level 1 and two followers at level 2. The system is shown in Fig. 3. The vector of decision variables is $\mathbf{x} = \left(x_1^1, x_1^2, x_2^2\right)$. The leader $\mathcal{D}_1^1$ controls x_1^1 and the followers $\mathcal{D}_1^2, \mathcal{D}_2^2$ control x_1^2, x_2^2 respectively. The objective functions of the players $f_1^1, f_1^2, f_2^2 : \mathbb{R}^3 \mapsto \mathbb{R}$ are all linear and the nonnegative decision variables must satisfy a set of five linear constraints $\mathbf{g} : \mathbb{R}^3 \mapsto \mathbb{R}^5$.

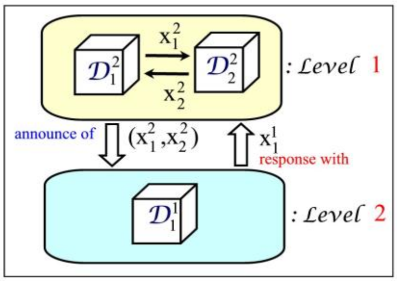

Figure 3 Decision making system for Example 2

B. Linear Bilevel Programming

The feasible region for this problem is shown in Fig. 4.

C. Nash-Stackelberg Equilibrium

The best responses BR_1^2 and BR_2^2 of the two followers are the heavy lines [A-E-F-G-C] and [D-A-B] respectively in Fig. 4. These lines are deduced from TABLE 2. The best responses of agent $\mathcal{D}_1^2$ for each possible choice of x_2^2 by $\mathcal{D}_2^2$ such that $x_2^2 \in \{0, 0.5, 1, 1.5, 2\}$ are calculated as follows. If . $\mathcal{D}_2^2$ chooses $x_2^2 = 0$, the response of $\mathcal{D}_1^2$ is $x_1^2 = 1$ (this is the only possibility) and the subsequent response of the leader $\mathcal{D}_1^1$ to these choices $\left(x_1^2, x_2^2\right) = (1, 0)$ is $x_1^1 = 0$. If $\mathcal{D}_2^2$ chooses $x_2^2 = 1$ (see highlighted numbers in TABLE 2), the response of $\mathcal{D}_1^2$ is multiple with $x_1^2 \in \{0, 0.5, \mathbf{1.5}, 2\}$. The corresponding payoffs for $\mathcal{D}_1^2$ being $f_1^2 \in \{-0.6, 0.75, \mathbf{1.45}, 0.8\}$, this player will retain the maximum and then chooses $x_1^{2*} = 1.5$. The subsequent response of the leader $\mathcal{D}_1^1$ to these choices $\left(x_1^2, x_2^2\right) = (1.5, 1)$ is $x_1^1 = 0.5$. The calculation goes on by using the same rules.

TABLE 2 DATA FOR EXAMPLE 2

Point	Coordinates			Function values		
	x_1^1	x_1^2	x_2^2	f_1^1	f_1^2	f_2^2
A	0	1	0	0.8	0.7	2
B	0	2	1	2.8	0.8	2.5
C	0	1	2	3.2	-0.5	-1
D	0	0	1	1.2	-0.6	-1.5
E	0.5	1	0.5	1.9	1.4	0.25

F	0.5	1.5	1	2.9	1.45	0.5
G	0.5	1	1.5	3.1	0.8	-1.25
H	0.5	0.5	1	2.1	0.75	-1.5

The point $A \in \left\{ BR_1^2 \cap BR_2^2 \right\}$. Is a Nash equilibrium. Since it is an element of S^2, it is also a Stackelberg solution.

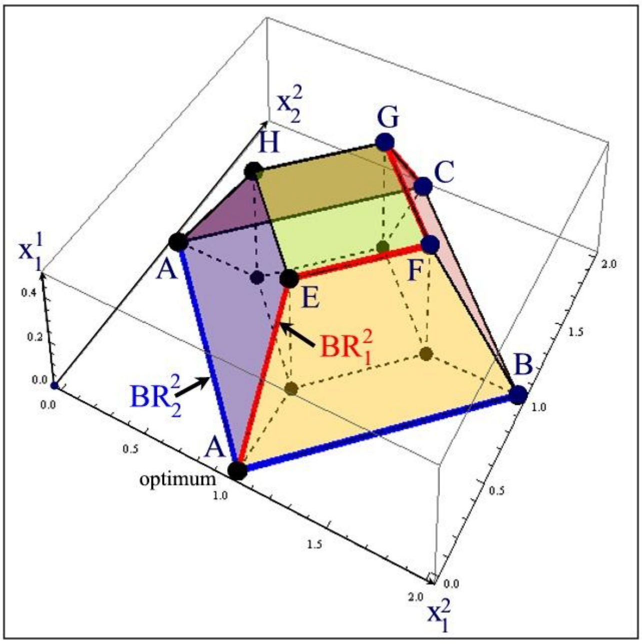

Figure 4 Nash-Stackelberg equilibrium for Example 2

This example is solved by using a 6 steps enumeration procedure in Yang and Bialas [31].

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