Partial Differential Equations to Diffusion-Based Population and Innovation Models

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I. Diffusion Process Modeling

I.1. Diffusion equation

- The one-dimensional diffusion equation is the Cauchy problem:

$$\partial_t N = \mathcal{D}\,\partial_x^2 N,\ N(x,0) = N_0(x), t \in (0,\infty),$$

where the state $N(x,t)$ is the density of population at time $t \in [0,\infty)$ and position $x \in \Omega \subset \mathbb{R}$, and $\mathcal{D}$ the diffusion coefficient.

- This parabolic PDE may be considered as the limiting form for the random motion by equal steps of a particle along the real line.
- The Fourier transforms and the convolution theorem of Fourier yields the solution:

$$N(x,t) = \frac{1}{2\sqrt{\mathcal{D}\pi t}} \int_{-\infty}^{\infty} N_0(v)\, e^{-\frac{(x-v)^2}{4\mathcal{D}t}}\, dv$$

I.2 Reaction-Diffusion Equations

- The scalar reaction-diffusion equation is

$$\partial_t N = R(N) + \mathcal{D}\, \partial_x^2 N$$

where the reaction rate takes one of the following specifications among others: (1) exponential growth, (2) logistic growth, (3) negative logistic growth, (4) asymmetric Gompertz.

$$R(N) = \begin{cases} rN & (1) \\ rN(1 - N/K) & (2) \\ -g^2 N(1 - N/K) & (3) \\ rN \ln(K/N) & (4) \end{cases}$$

- RD equation with exponential growth :

$$\partial_t N = rN + \mathcal{D}\, \partial_x^2 N$$

with the initial condition $N(x,0) = N_0(x),\ x \in (0,L)$

and the boundary conditions $N(0,t) = N(L,t) = 0$.

- The resolution of this IBVP yields

$$N(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{\left\{r - D\left(\frac{n\pi}{L}\right)^2\right\} t}$$

where $B_n = \frac{2}{L}\int_0^t N_0(s) \sin\frac{n\pi s}{L} ds$.

- The reaction rate increases the density locally and fasters the spatial spread in the population.

I.3 Delay Reaction-Diffusion Equations

- The temporal Wazewska-Czyzeska & Lasota delay RD equation (survival of red blood cells in animals) is :

$$\partial_t u = a(t)\,\partial_x^2 u - q(t)\,u(x,t-\tau),\ \tau > 0$$

- The spatio-temporal delay RD equation by Zhang & Zhou (2007) is:

$$\partial_t p = d\,\partial_x^2 p - \delta\, p(x,t) + q e^{-ap(x,t-\tau)}, \tau > 0, (x,t) \in \Omega \times (0,\infty)$$

where the state variable $p(x,t)$ denotes the number of red blood cells located at x at time t, δ a death rate.

II. Population Dispersal Models

II.1 Fisher-KPP Equation

- The one spatial dimensional Fisher-KPP equation is the parabolic PDE:

$$\partial_t N = rN(1-N) + \mathcal{D}\,\partial_x^2 N,\ x \in \Omega \subset \mathbb{R}$$

where $N(x,t)$ is for the population density at spatial position x at time t.

- The Fisher's equation with a logistic reaction term originally described the spreading of biological populations (e.g. the simulation of the propagation of a gene in a population (R.A. Fisher, 1930)).
- An RD equation such as Fisher-KPP for population models admits two main properties: 1. The solution is traveling the spatial domain at a finite rate of speed (**the traveling wave solutions**); 2. Conditions on the spatial domain are determined for a population persistence (**the critical patch size**).

- Generalized scalar RD equation to other non-population applications

$$\partial_t u = R(u) + \mathcal{D}\,\partial_x^2 u$$

- The reaction rate takes one of the following specifications: (1) Newell-Whitehead-Segel equation to describe a Rayleigh-Benard convection, (2) Zeldovich equation in combustion theory, (3) the degenerate Zeldovich equation, (4) the Kaliappan's generalization.

$$R(u) = \begin{cases} u(1-u^2) & (1) \\ u(1-u)(u-\alpha), \alpha \in (0,1) & (2) \\ u^2 - u^3 & (3) \\ u - u^k & (4) \end{cases}$$

- The generalized vector form of RD equations

is:

$$\partial_t \mathbf{u} = R(\mathbf{u}) + \mathcal{D}\nabla^2 \mathbf{u}$$

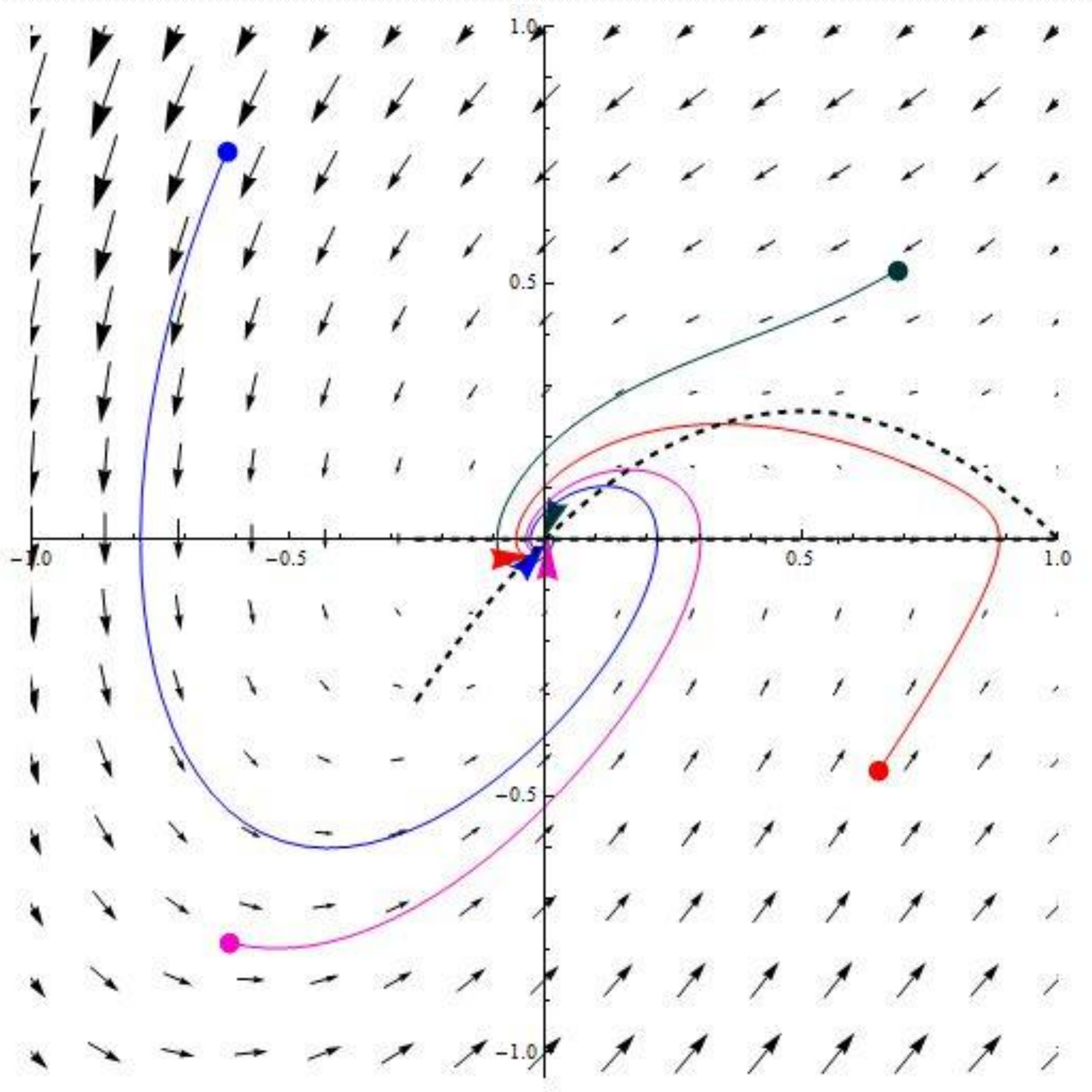
In chemistry, each component of $\mathbf{u}(\mathbf{x},t)$ represents the concentration of one substance. The semi-linear PDEs explain how the concentration of the substances in space changes due to two types of processes: local chemical reactions and diffusions of the substances in space.

II.2 Traveling Wave Solution

- **Definition** : For every wave speed $v \geq 2$, the Fisher's equation admits traveling wave solutions of the form $N(x,t) = F(x - vt) \equiv F(z)$ if traveling to the right, and $N(x,t) = G(x + vt)$ if traveling to the left. The function F is increasing and $\lim_{z \to -\infty} F(z) = 0$, $\lim_{z \to \infty} F(z) = 1$.
- The solution switches from one equilibrium state $N = 0$ to one another $N = 1$.
- For the special wave speed $v = \pm 5/\sqrt{6}$, all solutions are of the form:

$$F(z) = \left(1 + Ce^{\pm z/\sqrt{6}}\right)^{-2}, \quad C > 0$$

where $z = x \pm \frac{5}{\sqrt{6}} t$.



1.0
0.5
−1.0
−0.5
0.5
1.0
−0.5
−1.0

II.3 Critical Patch Size

- What is the minimal size of the spatial domain needed for a population survival ?
- For an RD with exponential growth which IBVP is $\partial_t N = rN + \mathcal{D}\,\partial_x^2 N,\ x \in (0, L)$ with the homogenous Dirichlet BCs $N(0,t) = N(L,t) = 0$ and $N(x,0) = N_0(x)$, the condition on the spatial domain so that the solution approach zero is $r < \mathcal{D}\left(\frac{\pi}{L}\right)^2$.
- Solving the equality for L yields the critical patch size $L_c = \pi\sqrt{\frac{\mathcal{D}}{r}}$.
- Thus the population size increases if $L > L_c$, and decreases if $L < L_c$.

III. Innovation diffusion models

III.1 Bass' Innovation Diffusion Model

- A marketing problem of new product acceptance: describe the process by which innovation products are communicated over time and expand through a population of adopters. How many of the potential aopters will buy the new product at time t ?
- The typical time path is a sigmoidal S-shaped time curve: few adopters at the beginning, then more and more adopters, and finally diffusion to public at large.
- The simplest diffusion model is:

$$\frac{dN}{dt} = g(t)\left(m - N(t)\right),$$

where $N(t)$ is the cumulative number of prir adopters, $m - N(t)$ the potential adopters and $g(t)$ the diffusion coefficient.

- A linear specification of the diffusion coefficient is :

$$g(t) = p + q\frac{N(t)}{m},$$

where m is the maximum of potential consumers, p and q , two control parameters, respectively the innovation and the imitation rates.

- Then, the Bass' model is the logistic ODE

$$\frac{dX}{dt} = (p + qX)(1 - X),$$

where $X(t) = N(t)/m$.

- Solving the Bass' model yields the time path:

$$X(t) = \frac{1 - e^{-(p+q)t}}{1 + (q/p)e^{-(p+q)t}}$$

- Then, the maximum diffusion rate is $\hat{X}(t) = \frac{1}{2} - \frac{p}{2q}$ and the corresponding time $\hat{t} = -\frac{\ln(p/q)}{p+q}$.

III.2 Stochastic Innovation Diffusion Model

- Random impacts from the environment (e.g. socioeconomic factors) as well from inside of the system.
- The modeling then consists of normally distributed parameters or formulating an Itô SDE. The reformulation by Skiadas & Giovanis (1997) is

$$dN = \left(p(m-N) + \frac{q}{m}(m-N)N \right) dt + c\left(\frac{p}{q} + \frac{N}{m} \right) dW$$

where W is a Wiener process and c the noise parameter.

- The expected solution is:

$$E[N] = \frac{m\, e^{(p+q)t}}{\dfrac{1}{\dfrac{p}{q} + \dfrac{N_0}{m}} + \dfrac{q}{p+q}\left(e^{(p+q)t} - 1 \right)} - \frac{mp}{q}$$

III.3 Spatial Innovation Diffusion Model with PDEs

- How innovations are diffusing in different geographical spaces ?
- Mahajan & Peterson (1979) integrate the space and time dimensions in the diffusion process. The Bass' innovation diffusion model becomes the following PDE

$$\partial_t N = \left(p(x) + q(x)N\right)\left(m(x) - N\right)$$

where $N(x,t)$ denotes the cumulative number of adopters in the domain x at time t.

- The innovation dynamics shows a characteristic wavelike set of S-shaped curves.

Thank you
for your attention !