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Stochastic Differential Games and Queueing Models To Innovation and Patenting

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Abstract. Dynamic differential games have been widely applied to the timing of product and device innovations. Uncertainty is also inherent in the process of technological innovation: R&D expenditures will be engaged in an unforeseeable environment and possibly lead to innovations after a random time interval. Reinganum [Reinganum, 1982] enumerates such uncertainties and risks: feasibility, delays in the process, imitation by rivals. Uncertainties generally affect the fundamentals of the standard differential game problem: discounted profit functional, differential state equations of the system, initial states. Two ways of resolution may be taken [Dockner, 2000]: firstly, stochastic differential games with Wiener process and secondly differential games with deterministic stages between random jumps (Poisson driven probabilities) of the modes. The player will then maximize the expected flows of his discounted profits subject to the stochastic state constraints of the system. In this context, the state evolution is described by a stochastic differential equation SDE (the Ito equation or the Kolmogorov forward equation KFE). According to the Dasguspta and Stiglitz's model [Dasguspta, 1980], R&D efforts exert direct and induced influences (through accumulated knowledge) about the chances of success of innovations. The incentive to innovate and the R&D competition can be supplemented by a competition around a patent. This presentation is focused on such essential economic and managerial problems (R&D investments by firms, innovation process , and patent protection) with uncertainties using stochastic differential games [Friedman, 2004], [Yeung, 2006], [Kythe, 2003], modeling with It SDEs [Allen, 2007] and queueing models [Gross, 1998]. The computations are carried out using the software *Mathematica* 5.1 and other specialized packages [Wolfram, 2003], [Kythe, 2003].

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1. Uncertainties and basic stochastic processes

Definition 1. *(stochastic process). Given a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with filtration* [2] $\{\mathcal{T}_t, t \geq 0\}$ *satisfying the conditions of right continuity and completion, a stochastic process is a collection of random variables* $\{x_t\}_{t \in T}$ *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* $\mathbb{R}^n$. *The process may be represented by the function [Øksendal, 2003]*

$$(t, \omega) \mapsto x(t, \omega) : T \times \Omega \mapsto \mathbb{R}^n.$$

We will thus have the number of random events that occur in [0,t].

Definition 2. *(random variable, path). For each* $t \in T$ *fixed, the state of the process is given. We then have a random variable (RV)defined by*

$$\omega \mapsto x_t(\omega), \omega \in \Omega.$$

For each experiment $\omega \in \Omega$, *we have a path of* x_t *defined by*

$$t \mapsto x_t(\omega), t \in T.$$

Definition 3. *(stationary process). A process* x_t *is stationary if* $x_{t_1}, \dots, x_{t_n}$ *and* $x_{t_{1+s}}, \dots, x_{t_{n+s}}$ *have the same joint distribution for all n and s.*[3]

1.1. Brownian motion

Definition 4 *(one-dimensional Brownian motion, or Wiener process [Friedman, 2004]. A stochastic process* $\{z_t\}_{t \geq 0}$, *is a Brownian motion satisfying the conditions:* (i) $z_0 = 0$, (ii) *the process has stationary independent increments* $z_{t_k} - z_{t_{k-1}} (1 \leq k \leq n)$, *and* (iii) *if* $0 \leq s < t$, $z_t - z_s$ *is normally distributed with* $\mathbf{E}[z_t - z_s] = (t - s)\mu$ *and* $\mathbf{E}[\{z_t - z_s\}^2] = (t - s)\sigma^2$, *where* μ *is the drift and* σ^2 *– the variance.*

According to the first condition any z_t that starts at z_0 can be redefined as $z_t - z_0$. The second condition tells that the random increment $z_{t_{n+1}} - z_{t_n}$ is independent of the previous one $z_{t_{n-1}} - z_{t_{n-2}}$, for all n. Increments are stationary when $z_t - z_{t-s}$ has the same distribution for any t and s constants. With the third condition, the RV has

[2] In algebra, a filtration of a group is ordinarily a sequence $G_n (n \in \mathbb{N})$ of subgroups such that $G_{n+1} \subseteq G_n$. A filtration is often used to represent the change of the sets of measurable events in terms of information quantity. A filtered σ-algebra is an increasing sequence of Borel σ-algebras $\{\mathcal{F}_t\}_{t \geq 0}$ with $\mathcal{F}_t \subseteq \mathcal{F}$ for each t and $t_1 \leq t_2 \Rightarrow \mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$.

[3] A weaker concept states that the first two moments are the same for all t and that the covariance between x_{t_1} and x_{t_2} depends on the time interval $t_1 - t_2$ [Ross, 1996].

the following probability density function (PDF) $f(z) = (\sqrt{2\pi\sigma^2 t})^{-1}\exp[-\frac{1}{2}(\frac{z-\mu}{\sigma})^2]$. Since the variance linearly increases in time, the Wiener process is non stationary. Stochastic differential equations (SDEs) frequently introduce uncertainty through a simple Brownian motion and are defined by

$$dx_t = \mu dt + \sigma dz_t,$$

where the constant μ is the drift rate, σ^2 the variance rate of x_t (σ denotes the diffusion rate), dt a short time interval, and dz_t the increment of the Brownian motion. The Figure 1 (a) shows three different realizations $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$ of the Brownian process. The deterministic part is clearly governed by the ordinary differential equation (ODE) $\dot{y}_t = \mu$ which solution is linear in time.[4]

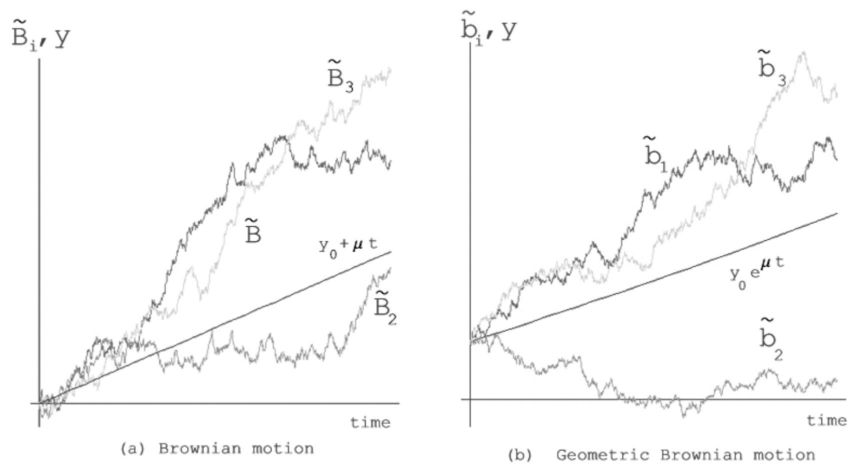

Fig. 1: Standard Brownian motions

Example (total factor productivity). Let us consider a differential representation for the production technology [Wälde, 2006]. With an AK technology, we have $Y_t = F(A_t, K) = A_t K$, where Y_t and A_t denote continuous functions of time t (a more convenient notation), where A states for the total factor productivity (TFP), Y, the

[4] The calculations use the packages Statistics‘ ContinuousDistributions, StochasticEquations‘ EulerSimulate of *Mathematica* 5.1 and Itovns3 [Kendall, 1993].The primitive of EulerSimulate is EulerSimulate [drift, diffusion, {x, x0}, {duration, nsteps}]. It returns a list of simulated values for the corresponding Ito process. More generally , a system of Ito processes can also be simulated by specifying the drift vector and the matrix of diffusion. The Mathematica package ItosLemma implements Ito's lemma for stochastic multidimensional calculus, computing stochastic derivatives and Ito–Taylor series. The primitive Itomake[x[t], μ, σ], where μ is the drift and σ the diffusion, stores the rule $x[t + dt] = x[t] + \mu dt + \sigma dB_1$.

output and K, a fixed amount of global productive factors. Suppose TFP grows at a deterministic rate g with $\dot{A}_t = gA_t \Leftrightarrow dA_t = gA_t dt$. We have the differential $dF(A_t, K) = F_A dA_t + F_K dK$.

We easily deduce the growth of Y_t as $\dot{Y}_t = gKA_t$. The evolution in time is

$$Y_t = gK \int_1^t A_s ds.$$

A more realistic situation consists in the introduction of the uncertainties that may affect the TFP. Let suppose that A_t will be driven by a Brownian motion with drift such as $dA_t = gdt + \sigma dz_t$, where g and σ are constants. Solving the SDE, we have $A_t = A_0 + gt + \sigma z_t$. The time evolution of Y_t is given by

$$Y_t = A_0 K + gKt + \sigma K z_t.$$

In this example[5], the evolution of the output consists of two parts: a deterministic trend and a stochastic deviation component from the trend. However, since Y_t may be negative, we have to look for another specification.

A RV may also evolve according to a geometric Brownian, such as

$$\frac{dx_t}{x_t} = \mu dt + \sigma dz_t \Leftrightarrow dx_t = \mu x_t dt + \sigma x_t dz_t.$$

The Figure 1 (b) shows three different realizations $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3$ of such a geometric Brownian process. The deterministic part is governed by the ODE $\dot{Y}_t = aY_t$, which solution is clearly exponential in time.

Definition 5. *(one-dimensional Itô processes [Øksendal, 2003]). Let B_t be a one-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A stochastic Itô integral is a stochastic process x_t of the form:*

$$x_t = x_0 + \int_0^t u_s(\omega) ds + \int_0^t v_s(\omega) dB_s,$$

so that

$$\mathbb{P}\{\textstyle\int_0^t v_s(\omega)^2 ds < \infty \text{ for all } t \geq 0\} = 1$$
$$\text{and } \mathbb{P}\{\int_0^t |u_s(\omega)| ds < \infty \text{ for all } t \geq 0\} = 1.$$

[5] Wälde [Wälde, 2006] also gives another specification where the drift rate is AK and the diffusion rate σK. The solution takes the form

$$Y_t = Y_0 + AKt + \sigma K z_t.$$

Theorem 1 *(the one-dimensional Itô formula [Øksendal, 2003]). Let x_t be an Itô process given by $dx_t = udt + vdB_t$. Let a twice continuously differentiable function $g(t,x) \in \mathcal{C}^2([0,\infty) \times \mathbb{R})$ on $[0,\infty) \times \mathbb{R}$. Then $y_t = g(t,x_t)$ is again an Itô process, and*

$$dy_t = \frac{\partial g}{\partial t}(t,x_t)dt + \frac{\partial g}{\partial x}(t,x_t)dx_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x_t)(dx_t)^2,$$

where $(dx_t)^2 = (dx_t).(dx_t)$ is calculated according to the multiplication rules

$$dt.dt = dt.dB_t = dB_t dt = 0, \; dB_t.dB_t = dt.$$

Proof.

See Øksendal [Øksendal, 2003], p.46 (with slightly different notations).

Example (Total factor productivity (continued) [Wălde, 2006]). Let TFP follow a geometric [6] Brownian motion

$$\frac{dA_t}{A_t} = gdt + \sigma dz_t,$$

where g and σ are constants. Hence, we have

$$\int_0^t \frac{dA_s}{A_s} = gt + \sigma z_t, \; z_0 = 0. \tag{1}$$

To evaluate the integral on the LHS, the Itô formula is used for the logarithmic function $g(t,x) = \ln x, \; x > 0$. We have

$$dg(t,x) = g_t dt + g_x dx + \frac{1}{2} g_{xx}(dx)^2,$$

where $dx = gxdt + \sigma x dz$. Since $(dx)^2 = g^2x^2(dt)^2 + \sigma^2x^2(dz)^2 + 2g\sigma dtdz = \sigma^2x^2dt$, we deduce

$$dg(t,x) = g_t dt + g_x dx + \frac{1}{2} g_{xx}\sigma^2 x^2 dt.$$

We then obtain

$$\begin{aligned} d\ln A_t &= \frac{1}{A_t}dA_t + \frac{1}{2}(-\frac{1}{A_t^2})(dA_t)^2 = \\ &= \frac{dA_t}{A_t} - \frac{1}{2A(t)^2}\sigma^2 A_t^2 dt = \frac{dA_t}{A_t} - \frac{1}{2}\sigma^2 dt. \end{aligned}$$

Hence, $\frac{dA_t}{A_t} = d\ln A_t + \frac{1}{2}\sigma^2 dt$. The expression of the integral is then

$$\int_0^t \frac{dA_s}{A_s} = \ln\frac{A_t}{A_0} + \frac{1}{2}\sigma^2 t.$$

[6] The process rather describes the rate of change of a RV, than the random variable itself.

From (1), we then deduce the time evolution of the TFP: $A_t = A_0 \exp[(g - \frac{1}{2}\sigma^2)t + +\sigma z_t]$.

1.2. Poisson Process

The occurrence of discrete events at times $t_0, t_1, t_2, \ldots$ (e.g. innovations) are often modeled as a Poisson process. For a Poisson process, the time intervals $\Delta t_1 = t_1 - t_0, \Delta t_2 = t_2 - t_1, \ldots$ between successive events are independent variables drawn from an exponential distributed population. The parameterized PDF is given by $f(x;\lambda) = \lambda \mathrm{e}^{-\lambda x}$ for some positive constant λ. Suppose a system that starts in state 0 at initial time t_0. It will change to state 1 at some time $t = T$, where T is drawn from an exponential distribution. The probability that the system will be in state 1 at time t_1 is given by the integral

$$P_1(t_1) = \int_0^{t_1} \lambda \mathrm{e}^{-\lambda t} \mathrm{dt} = 1 - \mathrm{e}^{-\lambda t_1}.$$

The probability of the system still being in state 0 is the complement $P_0(t_1) = \mathrm{e}^{-\lambda t_1}$. The absolute rate of change of being in state 1 is $\frac{dP_1}{dt} = \lambda \mathrm{e}^{-\lambda t}$. We deduce an exponential transition with rate λ such as

$$\frac{dP_1}{dt} = \lambda P_0. \tag{2}$$

More generally, for any number of states, a system of differential equations such as (2) will describe the probabilities of being in each state. Since the transition time from the state P_n to P_{n+1} is exponential for all n, a Poisson process will be deduced. Schematically, it is illustrated by the chain of the Figure 2, where P_j is the probability

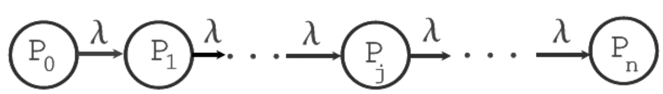

Fig. 2: Poisson process at constant rate

of the jth state when j events have occurred. The initial conditions are such that $P_0(0) = 1, P_j(0) = 0$ for all $j > 0$. We have to determine $P_n(t)$. Since the transitions are exponentially distributed, we have the system

$$\frac{dP_0}{dt} = -\lambda P_0,\ \frac{dP_1}{dt} = \lambda P_0 - \lambda P_1, \ldots,\ \frac{dP_n}{dt} = \lambda P_{n-1} - \lambda P_n.$$

Given the initial condition $P_0(0) = 1$, the solution of the first equation is $P_0(t) = \mathrm{e}^{-\lambda t}$. Substituting this result into the second equation, we have the ODE

$$\frac{dP_1}{dt} + \lambda P_1 = \lambda \mathrm{e}^{-\lambda t}.$$

Solving the ODE, we obtain [7]

$$P_1(t) = (\lambda t)\mathrm{e}^{-\lambda t}.$$

Continuing by substitution, we have

$$P_2(t) = \frac{(\lambda t)^2}{2!}\mathrm{e}^{-\lambda t}, \ldots, P_n(t) = \frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t}.$$

In this simple counting Poisson process, the probability $P_n(t)$ then expresses that exactly n events have occurred at time t. The expected number of occurrences by time t is

$$\mathbf{E}[n,t] = \sum_{n=0}^{\infty} nP_n(t) = \mathrm{e}^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!} = \mathrm{e}^{-\lambda t}(\lambda t)\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!} = \lambda t.$$

Definition 6 *(Poisson process). A stochastic process q_t is a Poisson process with arrival rate* [8] *λ if:* (i) $q_0 = 0$, (ii) *the process has independent increments, and* (iii) *the increments $q_\tau - q_t$ (or jumps) in any time interval $\tau - t$ is Poisson distributed with mean $\lambda(\tau - t)$, say $q_\tau - q_t \rightsquigarrow \mathcal{P}oi(\lambda(\tau - t))$.*

The probability that the process increases n times between t and $\tau > t$ is given by

$$\mathbb{P}\{q_\tau - q_t = n\} = \mathrm{e}^{-\lambda(\tau-t)}\frac{(\lambda(\tau-t))^n}{n!},\ n = 0, 1, \ldots.$$

The SDE of a standard Poisson process is

$$dx_t = adt + bdq_t,$$

where the increment dq_t is driven by

$$dq_t = \begin{cases} 0 & \text{w.p. } 1 - \lambda dt, \\ 1 & \text{w.p. } \lambda dt, \end{cases}$$

The Figure 3 illustrates two situations: in figure (a) jumps have the same amplitude, in the second (b) jump amplitudes are random. If no jump occurs ($dq = 0$), the variable will follow a linear growth $x_t = x_0 + at$.When a jump occurs, x_t increases by b. The Figure 3 (b) shows an extension of the Poisson process where the amplitude of the jumps b_t are governed by some distribution, such as $b_t \rightsquigarrow \mathcal{N}(\mu, \sigma^2)$.

[7] The solution of the generalized ODE (in usual notations) with variable parameters $\dot{x} + a(t) = b(t)$ is given by

$$x(t) = \mathrm{e}^{-\int_1^t a(s)ds}\{C + \int_1^t \mathrm{e}^{\int_1^s a(u)du} b(s)ds\},$$

where C is a constant of integration. The solution for the process is achieved by setting $x \equiv P_1$, $a(t) \equiv \lambda$ and $b(t) = \lambda\mathrm{e}^{-\lambda t}$.

[8] A high arrival rate means that the process jumps more often.

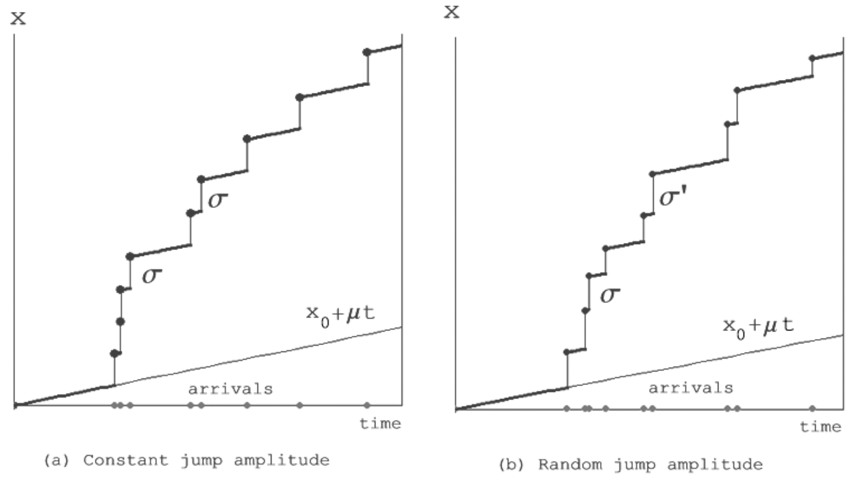

Fig. 3: Poisson processes with constant and random jump amplitude

Lemma 1 *(Change of Variable Formula CVF). Let x_t be a Poisson stochastic process given by*

$$dx_t = a(t, x_t, q_t)dt + b(t, x_t, q_t)dq_t.$$

Let a twice continuously differentiable function $F(t, x) \in \mathcal{C}^2([0, \infty) \times \mathbb{R})$ on $[0, \infty) \times \mathbb{R}$, the differential is

$$dF(t, x_t) = (F_t + F_x a(.))dt + \{F(t, x_{t^-} + b(.)) - F(t^-, x_t)\}dq_t,$$

where t^- denotes a date that precedes a jump at time t.

Example (Total factor productivity (continued) [Wälde, 2006]). Let TFP follow a geometric Poisson process

$$\frac{dA_t}{A_t} = gdt + \sigma dq_t,$$

where g and σ are constants. Hence, applying the Itô's lemma for Poisson processes, we obtain the solution [9]

$$A_t = A_0 \exp[gt + (q_t - q_0)\ln(1 + \sigma)].$$

The TFP will then follow a deterministic exponential trend and have a stochastic deviation, given by $(q_t - q_0)\ln(1 + \sigma), \sigma \geq 0$.

[9] The proof of lemma 1.9 is shown in appendix A. The detailed calculations for this example are given in the same appendix.

1.3. Queueing models

Queueing models are a typical application of exponential transitions and Poisson processes. Some events occur at some constant rate λ and are treated at a constant rate μ. Let us consider a M/M/1 queue, where the first M states for memoryless arrivals (i.e. inter-arrivals times of occurring events are often modeled as exponentially distributed variables), the second M is the same for departures and the 1 states a single server. The chain may by represented schematically by the Figure 4.

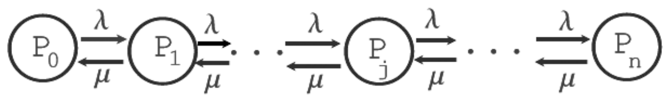

Fig. 4: Queues with a single service M/M/1

The system of dynamic equations is:

$$\frac{dP_0}{dt} = -\lambda P_0 + \mu P_1,\ \frac{dP_1}{dt} = \lambda P_0 - \lambda P_1 - \mu P_1 + \mu P_2, \dots,$$
$$\frac{dP_n}{dt} = \lambda P_{n-1} - \lambda P_n - \mu P_n + \mu P_{n+1}.$$

The steady state probabilities, once the system has stabilized at the equilibrium, are characterized by the probability that exactly n events occur and by the expected number of presently waiting events[10]

$$P_n = (1 - \frac{\lambda}{\mu})(\frac{\lambda}{\mu})^n \text{ and } \mathbf{E}[n] = \sum_{n=0}^{\infty} nP_n = \frac{\lambda/\mu}{1 - \lambda/\mu}.$$

2. Stochastic control problem and differential games

There are two particular ways to introduce uncertainty in the differential games. The first way is based on piecewise deterministic processes, where the system switches

[10] In the steady state, the derivatives vanish and we deduce the geometric series

$$P_1 = \frac{\lambda}{\mu} P_0, P_1 = (\frac{\lambda}{\mu})^2 P_0, \dots, P_n = (\frac{\lambda}{\mu})^n P_0, \dots$$

which converges only if the rate of arrivals λ is less than the rate μ of processing. Under this condition we have

$$P_0\{1 + (\frac{\lambda}{\mu}) + (\frac{\lambda}{\mu})^2 + \dots + (\frac{\lambda}{\mu})^n + \dots\} = P_0(1 - \frac{\lambda}{\mu})^{-1} = 1.$$

The expression of P_n will then follow from $P_0 = 1 - (\lambda/\mu)$.

from one deterministic mode to another, at random jump times. The second way consists of introducing continuous stochastic noise processes[11].

2.1. Optimal control under uncertainty

The uncertainty may take the form of a Brownian motion (also Wiener process). This process will generally influence the evolution of the state variable. This evolution will be described by an SDE of the form [Dockner, 2000]:

$$dx_t = f(t, x_t, u_t)dt + \sigma(t, x_t, u_t)dw_t, x_0 \text{ given,} \tag{3}$$

where x_t denotes the n-dimensional vector of the states and u_t – an m-dimensional vector of controls. A k-dimensional Wiener process w_t is a continuous-time stochastic process, such as $w : [0, T) \times \Xi \mapsto \mathbb{R}^k$, where Ξ denotes the set of points ξ of possible realizations of the RV. The functions are such that $f : \Omega = \{(t, x, u)|t \in [0, T), x \in X, u \in U(t, x)\} \mapsto \mathbb{R}^n$ and $\sigma : \Omega \mapsto \mathbb{R}^{n\times k}$[12]. A solution x_t of the SDE (3) must satisfy the following integral equation

$$x_t = \int_0^t f(s, x_s, u_s)ds + \int_0^t \sigma(s, x_s, u_s)dw_s$$

for all ξ of $w(s, \xi)$ in a set of probability 1[13] footnote Any solution to the SDE is a stochastic process depending on the realizations of $\xi \in \Xi$. The correct notation is rather $x(t, \xi)$ than x(t) or x_t, showing that the value of x_t cannot be known, without knowing the realization of ξ (see [Dockner, 2000], p.228)..

Lemma 2 *(Ito's lemma). Suppose that x_t solves the SDE (3). Let $G : [0, T)\times X \mapsto \mathbb{R}$ be a function with continuous partial derivatives G_t, G_x, G_{xx}. The function $g(t) = G(t, x_t)$ will satisfy the SDE:*

$$\begin{aligned} dg(t) = \{&G_t(t, x_t) + G_x(t, x_t)f(t, x_t, u_t)+ \\ &+ \frac{1}{2}tr[G_{xx}(t, x_t)\sigma(t, x_t, u_t).\sigma(t, x_t, u_t)']\}dt+ \\ &+ G_x(t, x_t)\sigma(t, x_t, u_t)dw_t. \end{aligned}$$

[11] Memoryless Poisson models of patent race are associated with Dasgupta and Stiglitz [Dasguspta, 1980], Lee and Wilde [Lee, 1980], Loury [Loury, 1979], Reinganum [Reinganum, 2004], [**?**]. The probability to innovate and to obtain a patent depends on the current R&D investment. Reinganum [Reinganum, 1982] and Yeung and Petrosyan [Yeung, 2006] consider non cooperative and cooperative games.

[12] An entry σ_{ij} of this $n \times k$ matrix evaluates the direct impact of the jth component of the k-dimensional Wiener process on the evolution of the ith component of the n-dimensional state vector.

[13] The second integral is such as $\lim_{\delta\to 0} \sum_{l=1}^{L-1} \sigma(t_l, x_{t_l}, u_{t_l})\{w_{t_{l+1}} - w_{(t_l}\}$, where $0 = t_1 < t_2 < \cdots < t_L = t$ and $\delta = \max\{|t_{l+1} - t_l|, 1 \leq l \leq L - 1\}$

The stochastic control problem is given by

$$\begin{aligned} \max \mathbf{E}_{u(.)}[\int_0^T & F(t,x_t,u_t))\mathrm{e}^{-rt}\ dt + \mathrm{e}^{-rT}S(x_T)], \\ & \text{s.t.} \\ dx_t = \ & f(t,x_t,u_t)dt + \sigma(t,x_t,u_t)dw_t, \\ & x_0 \text{ given}, u_t \in U(t,x_t). \end{aligned} \tag{4}$$

The following optimality conditions are based on the Bellman equation (HJB).

Theorem 2 *(Optimality conditions). Let a function be defined as* $V : [0,T) \times X \mapsto \mathbb{R}$ *with continuous partial derivatives* V_t, V_x *and* V_{xx}. *Assume that* V *satisfies the HJB equation:*

$$\begin{aligned} rV(t,x_t) - V_t(t,x_t) = max\ \{F(t,x_t,u_t) + V_x(t,x_t)f(t,x_t,u_t) \\ + \frac{1}{2}tr[V_{xx}(t,x_t)\sigma(t,x_t,u_t).\sigma(t,x_t,u_t)']|u_t \in U(t,x_t)\}, \end{aligned} \tag{5}$$

for all $(t,x_t) \in [0,T) \times X$. *Let* $\Phi(t,x_t)$ *be the set of controls maximizing the RHS of (5) and* u_t *be a feasible control path with state trajectory* x_t *s.t.* $u_t \in \Phi(t,x_t)$ *holds a.s. for a.a.* $t \in [0,T)$*:*
(i) *if* $T < \infty$ *and if the boundary condition* $V(T,x) = S(x)$ *holds for all* $x \in X$ *then* $u_{(.)}$ *is an optimal control path;*
(ii) *if* $T{=}\infty$ *and if either* V *is bounded and* $r > 0$, *or* V *is bounded below with* $\lim_{t\to\infty} e^{-rt}\boldsymbol{E}_{u_{(.)}}V(t,x_t) \leq 0$ *holds, then u(.) is a catching up optimal control path.*

Proof. See Dockner et al. [Dockner, 2000], p.229-30, Yeung and Petrosyan [Yeung, 2006], p.16 with different notations.

2.2. Differential games with random process

Let us consider a N-players game. The control variable by the ith player is denoted by u_t^i at time t for $i \in \{1,2,\ldots,N\}$. The vector of controls by the opponents of player i will be $u_t^{-i} = \{u_t^1, u_t^2, \ldots, u_t^{i-1}, u_t^{i+1}, \ldots, u_t^N\}$. The controls are subject to the constraints $u_t^i \in U^i(t,x_t,u_t^{-i}) \subseteq \mathbb{R}^{m_i}$, where the $x_t \in X$ are the state variables of the system. The state equation for the game will then be given (by omitting the time index of arguments)

$$dx_t = f(t,x,u^1,\ldots,u^N)dt + \sigma(t,x,u^1,\ldots,u^N)dw_t,$$

where w_t is a k-dimensional Wiener process. The functions f and σ are both defined on $\Omega = \{(t,x,u^i,u^{-i})|t \in [0,T), x \in X,\ u^i \in U^i(t,x,u^{-i})\}$ with values in $\mathbb{R}^n$ and $\mathbb{R}^{n\times k}$ respectively. The objective of each player is to maximize the expectation of his discounted flow of payoffs

$$J^i(u^i_{(.)}) = \ \mathbf{E}_{u_{(.)}}\{\int_0^T F^i(t,x_t,u^i,u^{-i})\mathrm{e}^{-r_it}dt + \mathrm{e}^{-r_it}S^i(x_T)|x_0 \text{ given}\},$$

where F^i is a real-valued utility function defined on Ω, S^i a real-valued scrap value function defined on X, and r_i the discount rate.

2.3. Piecewise deterministic control problem

Let us consider an autonomous problem defined over an unbounded time interval $[0, \infty)$. The evolution of the system may be deterministic, except at certain jump times given by the finite set $\{T_1, T_2, \ldots, T_M\}$. At each of these dates, the system switches from one mode to another. The following description is inspired from Dockner et al. [Dockner, 2000]. Let $X \subseteq \mathbb{R}^n$ denote the state space and $x_t \in X$ the state at time t. The set of controls, when the current mode is $h \in M$, is given by $U(h, x_t) \subseteq \mathbb{R}^m$.

The motion is described by the differential equation $\dot{x}_t = f(h, x_t, u_t)$ where $f(h, ., .)$ maps $\Omega(h) = \{(x, u) | x \in X, u \in U(h, x)\}$ into $\mathbb{R}^n$. The instantaneous payoffs of the player consist of $F(h, x_t, u_t)$, a real-valued function defined on $\Omega(h)$, and the lump sum payoff $S_{hk}(x_t)$, when a jump occurs from mode h to mode $k, (k \neq h)$. The payoffs are discounted at the constant rate $r > 0$. The motion of the system mode is a continuous-time stochastic process $h : [0, \infty) \times \Xi \mapsto M$, where the set Ξ of points ξ represents realizations of some random variable. Thus, the event, that the mode is h at time t, is $\{\xi \in \Xi | h_t(\xi) = h\}$ and its probability is denoted by $\mathbb{P}\{h_t(\xi) = h\}$. The probability that the system switches from mode h to mode k during the time interval $(t, t + \Delta t]$ is proportional to the length of Δ. We have

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \mathbb{P}\{h_{t+\Delta} = k | h_t = h\} = q_{hk}(x_t, u_t)\},\ k \neq h,$$

where $q_{hk} : \Omega(h) \mapsto \mathbb{R}_+$. The risk neutral player seeks to maximize the expectation of the discounted payoff flow, conditional on the initial state and mode. Initially, we state

$$J(u_{(.)}) = \mathbf{E}_{u_{(.)}}[\int_0^\infty F(h_t, x_t, u_t) e^{-rt} dt + \\ + \sum_{l \in \mathbb{N}} S_{h_{T_l -} h_{T_l}}(x_{T_l}) e^{-rT_l} \; | x_0, h_0 \text{ given}],$$

where T_l denotes the lth jump and $h_{T_l -}$, the mode immediately before the switch. We also have

$$J(u_{(.)}) = \mathbf{E}_{u_{(.)}}[\int_0^\infty \{F(h_t, x_t, u_t) + \\ + \sum_{k \neq h_t} q_{h_t k}(x_t, u_t) S_{h_t k}(x_t)\} e^{-rt} dt \mid x_0, h_0 \text{ given}]. \quad (6)$$

Definition 7 *(feasible path). Given Ξ a set of points $\xi \in \Xi$ representing possible realizations of a random variable, the control path $u : [0, \infty) \times \Xi \mapsto \mathbb{R}^m$ is feasible* *for the stochastic control problem if the following conditions are satisfied:* (i) *it is non-anticipating,* (ii) *the constraints* $x_t \in X$ *and* $u(t) \in U(h_t, x_t)$ *are a.s. verified,* (iii) *the process* $(h_{(.)}, x_{(.)})$ *and the integral in (4) are well defined. The control path is optimal, if it is feasible and if* $J(\bar{u}_{(.)}) \geq J(u_{(.)})$ *for all feasible paths.*

Dockner et al. [Dockner, 2000] p.206–7, deduce the fundamental theorem.

Theorem 3 *(optimal control path). Let us consider the autonomous problem* [14] *and suppose the existence of a bounded function* $V : M \times X \mapsto \mathbb{R}$ *which have the following properties. The function* $V(h, x)$ *is continuously differentiable in* x *for all* $h \in M$ *and is such that (omitting the time argument) the HJB equation*

$$rV(h,x) = max\ \{F(h,x,u) + V_x(h,x)f(h,x,u) + \\ + \sum_{k \neq h} q_{hk}(x,u)[S_{hk}(x) + V(k,x) - V(h,x)] | u \in U(h,x)\}, \quad (7)$$

is verified for all $(h, x) \in M \times X$. *Let* $\Phi(h, x)$ *be the set of controls maximizing the RHS of (7). Then the control path* u_t *is optimal if the following conditions are satisfied: the control* $u_{(.)}$ *is feasible and* $u_{(.)} \in \Phi(h_{(.)}, x_{(.)})$ *holds a.s.*

Proof. See Dockner et al. [Dockner, 2000], p. 207.

2.4. Piecewise deterministic differential games

Let a N-players autonomous game be a piecewise deterministic differential game over an infinite time horizon. Denote by u_t^i the control value by player i and

$$u_t^{-i} = (u_t^1, \ldots, u_t^{i-1}, u_t^{i+1}, \ldots, u_t^N)$$

the vector of controls of the opponents of player i. The player's i set of feasible controls is

$$U^i(h_t, x_t, u_t^{-i}) \subseteq \mathbb{R}^{m_i},$$

when the system is in mode $h_t \in M$ and state $x_t \in X$. The objective functional of the ith player is given by

$$J^i(u^i_{(.)}) = \mathbf{E}_{u_{(.)}}[\int_0^\infty F^i(h_t, x_t, u_t)\mathrm{e}^{-r_i t} dt \\ + \sum_{l \in \mathbb{N}} S^i_{h_{\theta_l -}\, h_{T_l}} x_{T_l} \mathrm{e}^{-r_i T_l} \mid x_0, h_0 \text{ given}],$$

where $S^i_{h_{(T_l -}\, h_{T_l}}(x_{T_l})$ denotes the payoff received if a jump occurs at time T_l from $h_{T_l -}$ to h_{T_l}. Suppose as in [Dockner, 2000], that all players use a stationary Markov

[14] Dockner et al. [Dockner, 2000] also consider the non-autonomous problem.

strategy of the form $u_t^i = \phi^i(h_t, x_t)$ then the player i's optimal control problem is of the form

$$\max_{u^i_{(.)}} J^i_{\phi^{-i}}(u^i_{(.)}) = \mathbf{E}_{u_{(.)}}[\int_0^\infty F^i_{\phi^{-i}}(h_t, x_t, u_t^i)e^{-r_i t}dt + \sum_{l\in\mathbb{N}} S^i_{h_{T_l-}, h_{T_l}}(x_{T_l})e^{-r_i T_l} \mid x_0, h_0 \text{ given}],$$

$$\text{s.t.}$$

$$\dot{x}_t = f^i_{\phi^{-i}}(h_t, x_t, u_t^i),$$

$$x_0 \text{ given},\ u_t^i \in U^i_{\phi^{-i}}(h_t, x_t),$$

where the piecewise deterministic process $h_{(.)}$ is determined by the initial condition h_0 and the switching rates $q^i_{\phi^{-i},hk}(x_t, u_t^i)$. The functions $F^i_{\phi^{-i}}$ and $f^i_{\phi^{-i}}$ have the same pattern as

$$F^i_{\phi^{-i}}(h, x, u^i) = F^i(h, x, \phi^1(h, x), \ldots, \phi^{i-1}(h, x), u^i, \phi^{i+1}(h, x), \ldots, \phi^N(h, x)).$$

The functions $U^i_{\phi^{-i}}$ and $q^i_{\phi^{-i},hk}$ are defined by

$$U^i_{\phi^{-i}}(h, x) = U^i(h, x, \phi^1(h, x), \ldots, \phi^{i-1}(h, x), \phi^{i+1}(h, x), \ldots, \phi^N(h, x)),$$
$$q^i_{\phi^{-i},hk}(x, u^i) = q^i(x, \phi^1(h, x), \ldots, \phi^{i-1}(h, x), u^i, \phi^{i+1}(h, x), \ldots, \phi^N(h, x)).$$

Ddefinition 8 *(stationary Markov–Nash equilibrium). A N-tuple of functions $\phi^i : M \times X \mapsto \mathbb{R}^{m_i}, i = 1, \ldots, N$ is a stationary Markov-Nash equilibrium of the game $\Gamma(h_0, x_0)$ if an optimal control path $u^i_{(.)}$ exists for each player i. If $(\phi^1, \ldots, \phi^N)$ is a stationary Markov–Nash for all games $\Gamma(h, x)$, then it is sub-game perfect.*

Theorem 4 *(stationary Markov–Nash equilibrium). Let us consider a given N-tuple of functions $\phi^i : M \times X \mapsto \mathbb{R}^{m_i}, i = 1, \ldots, N$ and assume that the piecewise deterministic process defined by state motion*

$$\dot{x}_t = f(h_t, x_t, \phi^1(h_t, x_t), \ldots, \phi^N(h_t, x_t))$$

and the switching rates

$$q_{hk}(x_t, \phi^1(h_t, x_t), \ldots, \phi^N(h_t, x_t))$$

is well defined for all initial conditions $(h_0, x_0) = (h, x) \in M \times X$. Suppose the existence of N bounded functions $V^i : M \times X \mapsto \mathbb{R}, i = 1, \ldots, N$ such that $V^i(h, x)$ is continuously differentiable in x and such that the HJB equations

$$r_i V^i(h, x) = max\ \{F^i_{\phi^{-i}}(h, x, u^i) + V^i_x(h, x) f^i_{\phi^{-i}}(h, x, u^i) + \sum_{k\neq h} q^i_{\phi^{-i},hk}(h, u^i)(S^i_{hk}(x) + V^i(k, x) - V^i(h, x))|\ u^i \in U^i_{\phi^{-i}}(h, x)\}, \quad (8)$$

are satisfied for all $i = 1, N$ and all $(h, x) \in M \times X$. Denote by $\Phi^i(h, x)$ the set of all $u^i \in U^i_{\phi^{-i}}(h, x)$ which maximize the RHS of (8). If $\phi^i(h, x) \in \Phi^i(h, x)$ and all $(h, x) \in M \times X$ then $(\phi^1, \ldots, \phi^N)$ is a stationary Markov-Nash equilibrium. The equilibrium is sub-game perfect.

Proof. See Dockner et al.[Dockner, 2000], p.212.

3. A stochastic game of R&D competition

The following game is initially due to Reinganum [Reinganum, 1982] and has been detailed by Dockner et al. [Dockner, 2000]. In this game N firms have competing R&D projects. The dynamic game supposes that: *(i)* no firm knows in advance the

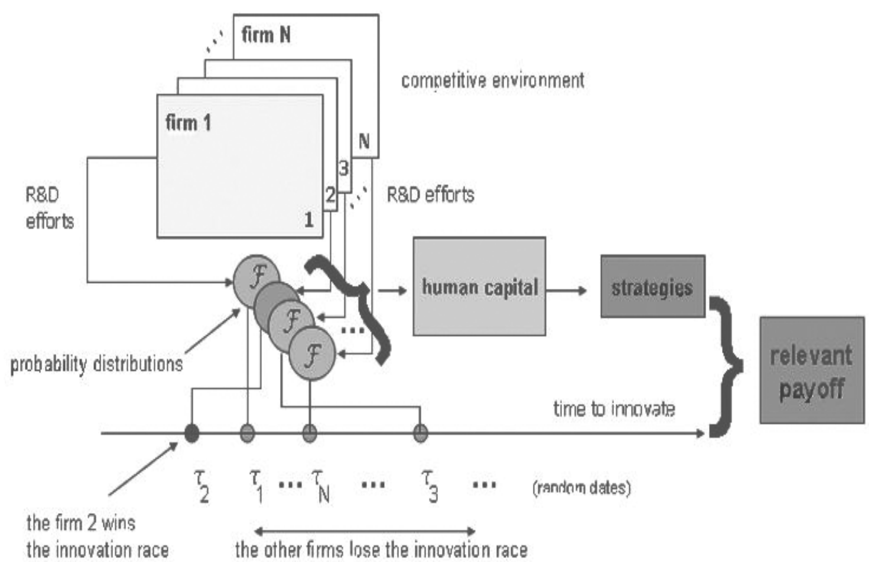

Fig. 5: A stochastic innovation game

amount of R&D that must be invested, *(ii)* R&D activities are costly but contribute to higher accumulation of common know-how and then have positive externalities *(iii)* one successful innovation may be achieved by using different paths. Resources in R&D positively influence the probability of successful innovations. Once a firm has won the competition, it acquires a monopolistic position. The problem is illustrated in the figure 5.

Game presentation

The time τ_i to complete a project is a RV, whose probability distribution is $F_{it} = \mathbb{P}\{\tau_i \leq t\}$, where for convenience F_{it} is taken similar to $F_i(t)$. The RVs τ_i are stochastically independent, since knowledge is supposed to have no spillover between firms. Denote by $\tau = \min\{\tau_i, i = 1, 2, \ldots, N\}$ the date of an innovation. According to independency, we may read

$$\mathbb{P}\{\tau \leq t\} = 1 - \prod_{i=1}^{N}(1 - F_{it}).$$

Let $u_{it} \geq 0$ be the rate of R&D effort. The rate of the distribution F_i is assumed to be proportional to the R&D efforts

$$\dot{F}_{it} = \lambda u_{it}(1 - F_{it}),\ F_{i0} = 0, \lambda > 0, \tag{9}$$

where $1 - F_i$ is the survival probability and $\dot{F}_{it}(1 - F_{it})^{-1}$ is the hazard rate. Assume that the present value of the innovator's net benefits P_I is constant and greater than the ones of the other competitors P_F. The costs of R&D efforts are quadratic in the investment rate. The game is played over a finite horizon T. This stochastic game belongs to the class of piecewise deterministic games. The system is in mode 0 before the innovation is happens. Ones an innovation by firm i has occurred, the system switches from one mode to another mode $i, i = 1, 2, \ldots, N$ [Dockner, 2000].

Game analysis

The expected discounted profit to be maximized by the ith player is

$$\int_0^T \{P_I \dot{F}_{it} \prod_{j \neq i}(1 - F_{jt}) + P_F \sum_{j \neq i} \dot{F}_{jt} \prod_{k \neq j}(1 - F_{kt}) - \\ - \frac{\mathrm{e}^{-rt}}{2} u_{it}^2 \prod_{j=1}^{N}(1 - F_{jt})\} dt. \tag{10}$$

This expression consists of three terms, which weights are the probabilities: firstly the firm i's value of net payoff P_I if the firm becomes the first to innovate, secondly the firm i's value of payoff P_F if this firm looses the competition and, thirdly, the discounted value of the cost of R&D efforts. Substituting the RHS of (9) into (10) the payoff expression is simplified as follows

$$\int_0^T \{\lambda P_I u_{it} + \lambda P_F \sum_{j \neq i} u_{jt} - \frac{\mathrm{e}^{-rt}}{2} u_{it}^2\} \prod_{j=1}^{N}(1 - F_{jt}) dt. \tag{11}$$

Let introduce the state transformation

$$-\ln(1 - F_{it}) = \lambda z_{it} \Leftrightarrow 1 - F_{it} = \mathrm{e}^{-\lambda z_{it}}.$$

The state variable z_{it} denotes the firm i's accumulated know-how via the R&D efforts. A differentiation w.r.t. time yields $\dot{F}_i(1-F_i)^{-1} = \lambda \dot{z}_i$. Hence, we have $\dot{z}_i = u_{it},\ z_{i0} = 0$. The corresponding payoff is deduced from (11)

$$J^i = \int_0^T \{\lambda P_I u_{it} + \lambda P_F \sum_{j \neq i} u_{jt} - \frac{\mathrm{e}^{-rt}}{2} u_{it}^2\} \times \exp[-\lambda \sum_{j=1}^N z_{jt}] dt.$$

Let y_t be equal to $\exp[-\lambda \sum_{j=1}^N z_{jt}]$. The differentiation w.r.t. time yields

$$\dot{y}_t = -\lambda y_t \sum_{j=1}^N u_{jt},\ y_0 = 1. \tag{12}$$

The game is then transformed to the following exponential game

$$J^i = \int_0^T \{\lambda P_I u_{it} + \lambda P_F \sum_{j \neq i} u_{jt} - \frac{\mathrm{e}^{-rt}}{2} u_{it}^2\} y_t dt,$$

$$s.t.$$

$$\dot{y}_t = -\lambda y_t \sum_{i=1}^N u_{it},\ y_0 = 1.$$

Markov–Nash equilibrium

When omitting the time argument, the current-value Hamiltonians ($i = 1, 2, \ldots, N$) are

$$\mathcal{H}^i(t, y, u_i \mu_i) = y\{\lambda P_I u_i + \lambda P_F \sum_{j \neq i} u_j - \frac{\mathrm{e}^{-rt}}{2} u_i^2\} - \mu_i \lambda y (u_i + \sum_{j \neq i} u_j),$$

where $\mu_i, i = 1, 2, \ldots, N$ are the current costate variables. The first order conditions (FOCs) are

$$\begin{aligned} \frac{\partial \mathcal{H}(t, y, u_i, \mu_i)}{\partial u_i} = 0 \Rightarrow u_i = \lambda \mathrm{e}^{rt}(P_I - \mu_{it}), \\ \dot{\mu}_i = -\frac{\partial \mathcal{H}^i(t, y, u_i, \mu_i)}{\partial y},\ \mu_{iT} = 0; \end{aligned} \tag{13}$$

$$u_{it} = -b_t \lambda \mathrm{e}^{rt}. \tag{14}$$

We have $N+1$ boundary conditions[15] from the state equation (12) and FOCs (13). To solve the boundary value problem let us conjecture a solution.

[15] There are $2N+1$ variables namely $y, \mu_i, u_i, i = 1, 2, \ldots, N$ and $2N+1$ equations.

Using the expression of u_i in (13) and substituting into (12) we have $\dot{y}_t = \lambda^2 N b_t y_t \mathrm{e}^{rt}$, $y_0 = 1$. To hold the conjectured solution, b_t must satisfy a Ricatti differential equation (RDE):

$$\begin{aligned} \dot{b}_t = -\frac{\lambda^2 \mathrm{e}^{rt}}{2}\{(2N-1)b_t^2 + 2b_t(1-N)(P_F - P_I)\}, \\ b_T = -P_I. \end{aligned} \tag{15}$$

The solution of the RDE (15) can be found by an CVF method letting $g(t) = -1/b_t$.

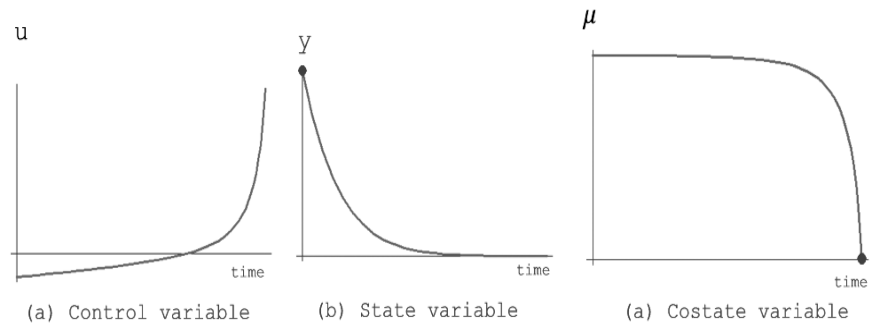

Fig. 6: Control, state, costate in an innovation game with two firms

Substituting the solution of b_t into (14) yields the Markovian identical strategies $u_t =$

$$= \frac{2\lambda P_I(P_I - P_F)(N-1)\mathrm{e}^{rt}}{(2N-1)P_I + \{P_I + 2(N-1)P_F\} \times \exp[\frac{1}{r}(P_I - P_F)(N-1)\lambda^2(\mathrm{e}^{rt} - \mathrm{e}^{rT})]}.$$

The Figure 6 shows the control variable, the state and the costate variables, when the game is reduced to two competitors ($N = 2$).

4. A stochastic game of patent race

In some market models N incumbents and one entrant aim to invent a new process or a new product, and patent their innovations. In the model of [Dasguspta, 1980], [Reinganum, 1982], [Tirole, 1990]the patent race is to develop a cost reducing production process for an existing product, in the model of Gayle [Gayle, 2001] the firm are patenting a new product. This model is in line with the one of Loury [Loury, 1979] and its developments by Lee and Wilde [Lee, 1980] where a stochastic relationship is assumed between R&D investments and the time at which an innovation will occur.

Description of the game

An output market is composed of $N+1$ firms, which are attempting to produce a new product and taking simultaneously a patent[16]. N of these firms are incumbents and one firm is a potential entrant. The rate of investment in R&D is $x_i, i=1,\ldots,N$ for an incumbent, and z for the entrant. The date of success for an innovation is supposed to depend only on the R&D investment rate[17] such as $\tau_i(x_i)$. The probability that the firm i succeeds at or before the date t is

$$\mathbb{P}\{\tau_i(x_i)\leq t\}=1-\exp[-h(x_i)t],\ t\in[0,\infty),$$

where $h(.)$ denotes the hazard function. Let this function be twice differentiable, strictly increasing, concave and satisfying the following conditions: $h'(.)>0, h''(.)<0$ for all $x_i, z\in[0,\infty),$ and $\lim_{x_i,z\to\infty}h'(.)=0$. The conditional probability that the firm i will succeed in the instant, given that it has not already succeeded is

$$\mathbb{P}\{\tau_i(x_i)\in(t,t+dt]|\tau_i>t\}=h(x_i)dt,\ t\in[0,\infty).$$

This result is due to the memoryless property of the exponential distribution[18]. Let $\bar{\tau}_i$ represents the date at which the first firm introduces an innovation. Then $\bar{\tau}_i=\min_{j\neq i}\{\bar{\tau}(x_j)\}$, and

$$\begin{aligned}\mathbb{P}\{\bar{\tau}_i\leq t\}&=1-\mathbb{P}\{\tau_j>t\},\ \text{for all } j\neq i\\&=1-\exp[\sum_{j\neq i}h(x_j)].\end{aligned}$$

Finally, let us suppose the following constant non-discounted profits: P_i^A the incumbent i's profit before any innovation occurs, P_i^W the incumbent i's profit if he wins the patent race, P_i^{LE} the incumbent i's profit if he loses the patent race to the entrant, P_i^{LI} the incumbent i's profit if he loses the patent race to another incumbent, and P_E the profit of the entrant when he wins the patent race.

The expected profits of incumbents i, $i=1,2,\ldots,N$ are given by:

$$\begin{aligned}V^i=\int_0^\infty &\exp[-\{h(x_i)+\sum_{k\neq i}^N h(x_k)+h(z)\}t]\times\\&\times\{P_i^A-x_i+h(x_i)\frac{P_i^W}{r}+h(z)\frac{P_i^{LE}}{r}+\sum_{k\neq i}^N h(x_k)\frac{P_i^{LI}}{r}\}\times \mathrm{e}^{-rt}dt. \quad (16)\end{aligned}$$

[16] The following game of patent race is inspired from the presentation of Gayle [Gayle, 2001].

[17] Reinganum [Reinganum, 1982] [**?**] assumes that the probability of introducing an innovation is a function of both the current of investment in R&D and the accumulated stock of technology.

[18] Let the PDF of the R.V. T be $f(t)dt=\lambda\mathrm{e}^{-\lambda t}dt$. We have the moment $\mathbf{E}[T^k]=\int_0^\infty t^k f(t)dt$. Then $\mathbf{E}[T^k]=\Gamma(k+1)\lambda^{-k}=\lambda^{-k}k!$, since $\Gamma(k)=\int_0^\infty \mathrm{e}^{-u}u^{k-1}du=(k-1)!$. Hence, $\mathbf{E}[\tau]=\lambda^{-1}$.

The expected profit of the entrant is

$$V^E = \int_0^\infty \exp[-\{h(x_i) + \sum_{k\neq i}^N h(x_k) + h(z)\}t] \times \{h(z)\frac{P_E}{r} - z\} \times \mathrm{e}^{-rt} dt. \qquad (17)$$

Integrating (16) and (17), we have the system for the R&D subgame

$$V^i = \frac{P_i^A - x_i + h(x_i)\frac{P_i^W}{r} + h(z)\frac{P_i^{LE}}{r} + \sum_{k\neq i}^N h(x_k)\frac{P_i^{LI}}{r}}{r + h(x_i) + \sum_{k\neq i}^N h(x_k) + h(z)},$$

$$V^E = \frac{h(z)\frac{P_E}{r} - z}{r + h(x_i) + \sum_{k\neq i}^N h(x_k) + h(z)}.$$

Best response functions

The reaction functions (BRFs) are then deduced from the following $N+1$ FOCs

$$\frac{\partial V^i}{\partial x_i} = 0,\ i = 1, \ldots, N \text{ and } \frac{\partial V^E}{\partial z} = 0.$$

A Nash equilibrium of R&D spending must then satisfy the following conditions (omitting the nonzero denominator of the LHS):

$$\{r + h(x_i) + \sum_{k\neq i}^N h(x_k) + h(z)\} \times (h'(x_i)\frac{P_i^W}{r} - 1) -$$
$$- \{P_i^A - x_i + h(x_i)\frac{P_i^W}{r} + h(z)\frac{P_i^{LE}}{r} + \sum_{k\neq i}^N h(x_k)\frac{P_i^{LI}}{r}\}h'(x_i) = 0, \qquad (18)$$

$$\{r + h(x_i) + \sum_{k\neq i}^N h(x_k) + h(z)\}(h'(z)\frac{P_E}{r} - 1) - (h(z)\frac{P_E}{r} - z)h'(z) = 0. \qquad (19)$$

The equation (18) is the incumbent i's BRF with upward sloping[19], given the R&D investments of his opponents. Similarly, (19) is the entrant's BRF with upward sloping, given the R&D investments of his opponents.

[19] In the case of symmetry between the BRFs of the incumbent and entrant, when $N = 1, PLE = PA = 0$, we have the incumbent's BRF

$$F(x, z) = \{h'(x) + \frac{1}{r}h'(x)h(z)\}P^W - h'(x)P^A - r - \{h(x) + h(z)\} + xh'(x).$$

By the implicit function theorem, we deduce

$$\frac{dx}{dz} = -\frac{F_z}{F_x} = -\frac{h'(z)\{h'(x)\frac{P^W}{r} - 1\}}{h''(x)\{1 + \frac{h(z)}{r}\}P^W + xh''(x)}.$$

The derivative dx/dz is positive because of a non-negative numerator with non-negative expected profits and due to a negative denominator with the concavity of h(.).

Symmetric Nash equilibrium

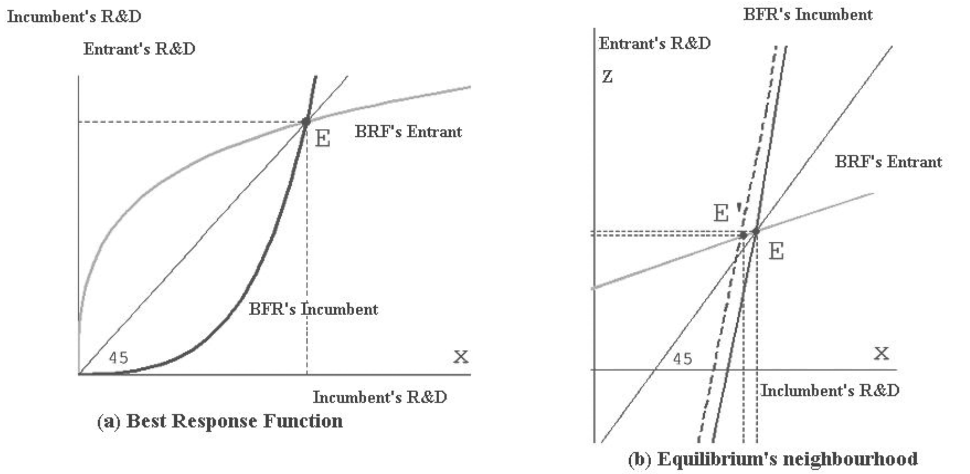

Fig. 7: Symmetric Nash equilibria

A symmetry of the BRFs is achieved when $N = 1, PLE = PA = 0$. The BRFs are then defined by

$$\{h'(x) + \frac{1}{r}h'(x)h(z)\}P_i^W - h'(x)P_i^N - r - \{h(x) + h(z)\} + xh'(x) = 0,$$
$$\{h'(z) + \frac{1}{r}h'(z)h(x)\}P_E - r - \{h(x) + h(z)\} + zh'(z) = 0.$$

The Figure 7 (a) shows the reaction functions and the Nash equilibrium. Due to the stability condition, the incumbent's reaction function is steeper than the entrant's one[20]Both reactions functions intersect on the 45° line. The expected pre-innovation profit P^A only occurs in the incumbent's BRF. A positive P^A will then shift the incumbent's BRF to the left, as it is shown in figure Fig.(b). As a consequence to the shift, we observe with [Gayle, 2001] that the R&D spending equilibrium of the incumbent will be less than that the one of the entrant.

5. Concluding remarks

[20] The stability condition is expressed by

$$|V^i_{xx}| > |V^i_{xz}| \text{ and } |V^E_{zz}| > |V^E_{zx}|$$

in the neighborhood of the equilibrium (see Figure 7 (b)).

The differential games in R&D economics give some indications and results about debates and questions, like the relation between the concentration of an industry and the intensity of R&D investments. The consequences of the market structures (socially managed market, pure monopolist with barriers to entry, competitive economy) on R&D have been studied notably by Dasguspta and Stiglitz [Dasguspta, 1980]. For example, the correlation between concentration and R&D efforts depends upon the existing degree of concentration and upon free entrance. For Loury [Loury, 1979], given a market structure, firms will rather more invest in R&D than is socially optimal. Reinganum [Reinganum, 1982] considers the optimal resource allocation in R&D, the social optimality of the game and implications of innovation policies (taxes and subsidies, patents). Noncooperation or cooperation aspects also play a great role. If costs are not too high, a firm may choose to cooperate in R&D investments with one or some other firms. Dasguspta and Stiglitz [Dasguspta, 1980], Reinganum [Reinganum, 1982], Amir, Evstigneev, Wooders [Amir, 2003] have studied such cooperations. Yeung and Petrosyan [Yeung, 2006] consider the theory of the cooperative stochastic differential games and analyze a cooperation R&D game under uncertainty over a finite planning period. This paper has introduced to the approach of stochastic games in theory and application. Two simple examples of noncooperative games in R&D have been developed. The first example of stochastic game belongs to the class of piecewise deterministic games. Then, when an innovation occurs, the system switches from one mode to another. Indeed, the winning firm is supposed to acquire immediately a monopolistic position. The second example considers a stochastic game between several incumbents of an industry and a potential entrant. In this model, the R&D spending equilibrium of the representative incumbent (the game is symmetric) is rather less than the one of the entrant. Recent papers relax or improve some strong assumptions. Sennewald [Sennewald, 2007] relaxes the strong assumption of boundedness conditions when applying the Bellman equation (bounded utility function, bounded coefficients in the differential equation). The HJB can then still be used with linearly boundedness. Let us also indicate the less recent model of endogenous growth by Aghion and Howitt [Aghion, 1992]. In this model, according to a forward-looking difference equation, the research expenses in any period depend upon the expected research investment next period.

Appendix

Ito's lemma for Poisson Processes

Ito's lemma for Poisson processes is not frequently presented in textbooks. This appendix shows the corresponding rule, its proof and one application to TFP.

Lemma 3 *(Ito's lemma for a Poisson process). Given a Poisson SDE*

$$dx_t = adt + bdq_t, \text{ where a and b are constant.} \tag{20}$$

Let $F(t, x_t)$ be a continuously differentiable equation of t and x. Then, we have

$$dF(t, x_t) = (F_t + aF_x)dt + \{F(t, x_t + b) - F(t, x_t)\}dq_t. \tag{21}$$

Proof. By differentiating the function $F(t, x_t)$ and using equation (20), we have

$$\begin{aligned} dF(t, x_t) &= F(t + dt, x_{t+dt}) - F(t, x_t) = \\ &= F(t + dt, x_t + adt + bdq_t) - F(t, x_t). \end{aligned} \tag{22}$$

Adding and subtracting $F(t + dt, x_t + adt)$ in equation (22) we have

$$\begin{aligned} dF(t, x_t) = F(t + dt, x_t + adt + bdq_t) - F(t + dt, x_t + adt) - \\ -F(t, x_t) + F(t + dt, x_t + adt). \end{aligned} \tag{23}$$

The last two terms of equation (23) correspond to a situation where we have no jump. Hence, with $x_t = at$ and $dx_t = adt$, we have

$$\begin{aligned} F(t + dt, x_t + dx_t) - F(t, x_t) &= F_t dt + F_x dx_t = \\ &= (F_t + aF_x)dt. \end{aligned} \tag{24}$$

According to equations (23) and (24), we deduce

$$\begin{aligned} dF(t, x_t) = F(t + dt, x_t + adt + bdq_t) - F(t + dt, x_t + adt) + \\ +(F_t + aF_x)dt. \end{aligned} \tag{25}$$

The first two terms of equation (25) have a different expression according that a jump may or not occur. We have

$$F(t + dt, x_t + adt + bdq_t) - F(t + dt, x_t + adt) =$$

$$\begin{aligned} &= \begin{cases} F(t, x_t + b) - F(t, x_t) & \text{w.p. } \lambda dt\text{, with jump} \\ 0 & \text{w.p. } 1 - \lambda dt\text{, without jump} \end{cases} \\ &= \{F(t, x_t + b) - F(t, x_t)\} dq_t. \end{aligned}$$

Then, we find the Ito's formula for a Poisson stochastic differential equation

$$dF(t, x_t) = (F_t + aF_x)dt + \{F(t, x_t + b) - F(t, x_t)\} dq_t.$$

Example (Total factor productivity). The stochastic differential equation is given by [Wälde, 2006])

$$\frac{dA_t}{A_t} = gdt + \sigma dq_t, \tag{26}$$

where g and σ are constants. The equation (26) is equivalent to

$$dA_t = gA_t dt + \sigma A_t dq_t.$$

In this example, the parameters of the stochastic differential equation (20) are $a \equiv gA_t = a_t$ and $b \equiv \sigma A_t = b_t$. Let us apply the Ito's formula (21). Taking $F(t, A_t) = \ln A_t$, we have $F_t = 0$ and $F_A = 1/A_t$. We also determine

$$\ln(A_t + \sigma A_t) - \ln A_t = \ln(1 + \sigma).$$

Then according to Ito's formula, we have

$$d \ln A_t = g + \ln(1 + \sigma) dq_t. \tag{27}$$

By integrating both sides of equation (27), we obtain

$$\int_0^t \ln A_s = gt + \ln(1 + \sigma) \int_0^t q_s.$$

Hence,

$$[\ln A_s]_0^t = gt + [q_s]_0^t,$$

and

$$\ln A_t = \ln A_0 + gt + \ln(1 + \sigma)(q_t - q_0).$$

The path of the total factor productivity is then given by

$$A_t = A_0 \exp[gt + \ln(1 + \sigma)(q_t - q_0)].$$

References

Aghion P., Howitt P. 1992. A Model of Growth through Creative Destruction. *Econometrica*, 60 (2): 323–351.

Allen E. 2007. Modeling with Itô Stochastic Diffrential Equations, Dordrecht. The Netherlands. Springer.

Amir R., Evstigneev, Wooders J., Noncooperative versus Cooperative R&D with Endogeneous Spillover Rates. *Games and Econ. Behav*, 42: 183–207.

Dasguspta P., Stiglitz J. 1980. Uncertainty, Industrial Structure and the Speed of R&D. *Bell J.Econom.*, 11: 1–28.

Dockner E., Jorgenson S., Long N.V.,Sorger G., 2000. Differential Games in Economics and Management Science. Cambridge. Mass., Cambridge University Press.

Friedman A., 2004 Stochastic Differential Equations and Applications. New York. Dover Publications Inc.

Gayle P.G., 2001 Market Structure and Product Innovation. Boulder (Colorado). University of Colorado; 01–15.

Gross D., Harris C.M. 1998. Fundamentals of Queueing Theory. ed. 3. New York. Wiley & Sons

Kendall W. S. 1993. Itovsn3: Doing Stochastic Calculus with Mathematica. In: H.R. Varian, Edit, *Economic and Financial Modeling with Mathematica.* New York. Springer-Verlag; 214–238.

Kythe P.K., Puri P., Schäferkotter M. 2003. Partial Differential Equations and Boundary Value Problems with Mathematica. ed. 2. New York. Chapman & Hall/CRC.

Lee T., Wilde L. 1980. Market Structure and Innovation : A Reformulation. *The Quart. J. Econ.*, 194: 429–436.

Loury G.C. 1979. Market Structure and Innovation, *The Quart. J. Econ.*, 93: 395–410.

Malliaris A.G., Brock W. 1984. Stochastic Methods in Economics and Finance. Amsterdam, New York, Oxford, North Holland.

Øksendal B. 2003 Stochastic Differential Equations : An Intoduction with Applications. Berlin Heidelberg. Springer-Verlag.

Reinganum J.F., 1982. A Dynamic Game of R and D : Patent Protection and Competitive Behavior. *Econometrica*, 50 (3): 671–688

Ross S. 1996. Stochastic Processes. ed.2. New York. John Wiley & Sons.

Sennewald K. 2007 Controlled Stochastic Differential Equations under Poisson Uncertainty and with Unbounded Utility. *J.Econ.Dynam.Control*, 31: 1106–1131.

Tirole J. 1990 The Theory of Industrial Organization. Cambridge. Mass. London. The MIT Press.

Wälde K. 2006 Applied Intertemporal Optimization. lecture notes. University of Würzburg. Germany. http://www.waelde.com/aio.

Wolfram S. 2003. The MATHEMATICA Book. ed. 5. Champain U. Wolfram Media Inc.

Yeung D.W.K., Petrosyan L. 2006, Cooperative Stochastic Differential Games. New York. Springer Science. Business Media Inc.