5th International Conference on
Applied Mathematics, Simulation, Modelling (ASM'11)

Corfu Island, Greece, July 14-16, 2011

Stochastic delay Lotka–Volterra system to interacting population dynamics

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1. Predator-prey models

- General system with 2 interacting populations

$$\frac{x_1(t)}{dt} = x_1(t)\big(b_1(.) - d_1(.)\big),$$

$$\frac{x_2(t)}{dt} = x_2(t)\big(b_2(.) - d_2(.)\big).$$

$$where\ (.) \equiv (x_1, x_2)$$

- Standard subclass LV model

$$LV: b_1(.) = b_1, d_1(.) = c_1 x_2, b_2(.) = c_2 x_1, d_1(.) = d_2.$$

Nonzero SS equilibrium at point (d2/c2,b1/c1) and solutions form closed curves in the (x1,x2)-plane to the SS equilibrium.

1. Predator-prey model (continued 1)

- Variant of the LV model with logistic growth

$$\frac{x_1(t)}{dt} = x_1(t)\left(max\{\frac{a_1(k_1 - x_1)}{k_1}, 0\} - c_1 x_2\right),$$

$b_1(.)$

$$\frac{x_2(t)}{dt} = x_2(t)(c_2 x_1 - d_2).$$

Nonzero SS equilibrium at point (d2/c2,(a1c2k1-a1d2)/c1c2k1) and solutions spiral in the (x1,x2)-plane to the SS equilibrium.

- Generalized LV model to *n*-species

$$\frac{dx_i(t)}{dt} = x_i(t)\left(a_i + \sum_{j=1}^{n} b_{ij}\, x_j(t)\right), i = 1, \dots, n$$

1. Predator–prey model (continued 2)

- ## Environmental variability
- One possible way to model is to introduce additional arguments into the b(.)'s per capita birth rate functions, as into the d(.)'s per capita death rate functions.
- ## Stochastic LV model

$$dx_1 = x_1\big(b_1(.) - d_1(.)\big)dt + \sqrt{x_1\big(b_1(.) + d_1(.)\big)}dW_1(t),$$

$$dx_2 = x_2\big(b_2(.) - d_2(.)\big)dt + \sqrt{x_2\big(b_2(.) + d_2(.)\big)}dW_2(t).$$

- The mean persistence–time of the system (i.e. the expected time it takes for the size of a population to reach zero) can be estimated through a numerical solution or by solving the backward Kolmogorov equation.

2. Delay L-V system

- An autonomous competitive or cooperative LV model with delays is of the form

$$\frac{dx_i(t)}{dt} = x_i(t)\left(b_i - \sum_{j=1}^{n} a_{ij}\, x_j(t) - \sum_{j=1}^{n} b_{ij}\, x_j(t-\tau_{ij})\right), \qquad i = 1, \dots, n$$

- The permanence supposes a positive solutin to the system

$$b_i - \sum_{j=1}^{n} a_{ij}\, x_j - \sum_{j=1}^{n} b_{ij}\, x_j = 0, \qquad i = 1, \dots, n$$

2. Delay L-V system (continued 1)

- Time delays in biological systems are a source of nonstationary problems (periodic oscillations, instabilities). The loss os stability intervenes at a certain threshold.

- However, time delays can enhance stability; short delays can also stabilize unstable dynamical systems.

2. Delay L-V system (continued 2)

- **Example:** Let an 2 species LV model with 2 delays. The biomass of the predator (or parasite) and prey (or host) are x1(t) and x2(t) respectively. The system is

$$\frac{dx_1(t)}{dt} = x_1(t)\big(-1 + x_2(t - \tau_2)\big)$$

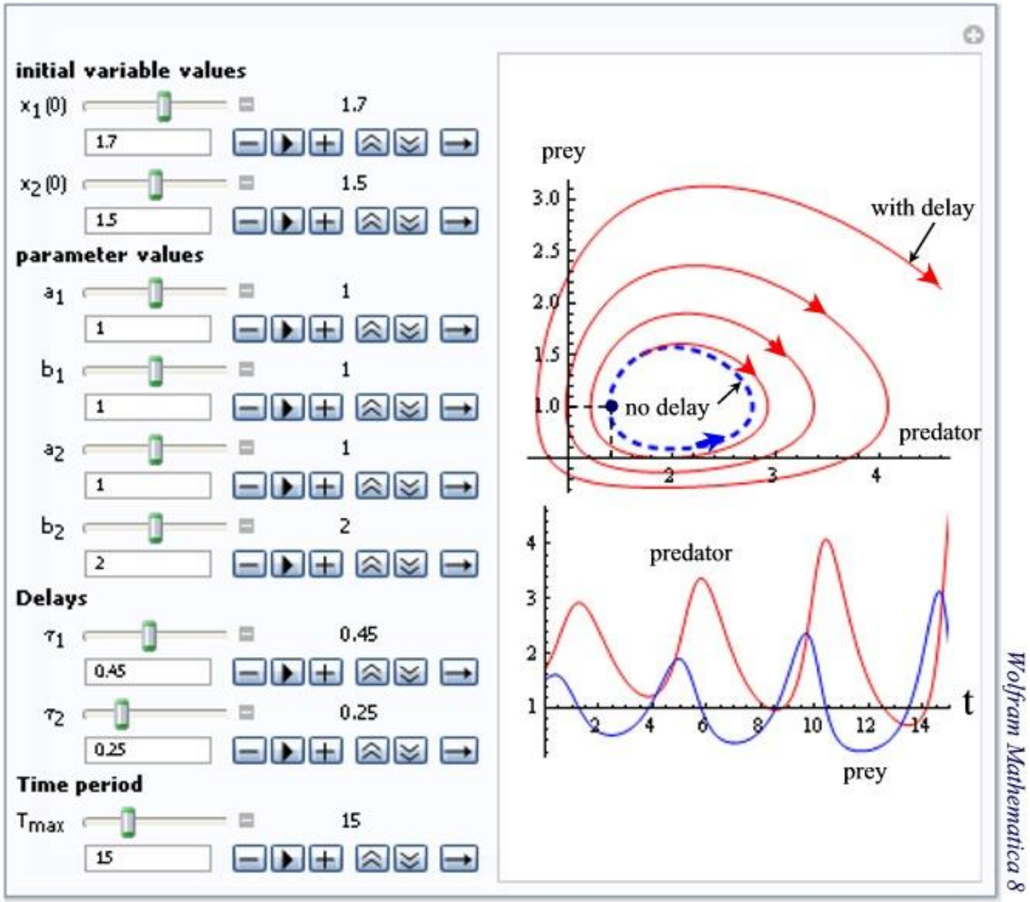
$$\frac{dx_2(t)}{dt} = x_2(t)\big(2 - x_1(t - \tau_1)\big)$$

- A stable periodic solution for the nondelay model is

$$H(t) \equiv 2\, ln\, x_1(t) + x_1(t) + ln\, x_2(t) - \; x_2(t) = k_1$$

2. Delay L-V system (continued 3)

- The interactive *Mathematica* object is



3. Stochastic delay L-V system

- **Nondelay stochastic LV model:** suppose that all the *b*'s parameters are stochastically perturbed with

$$b_{ij} \to b_{ij} + \sigma_{ij}\dot{w}(t)$$

The corresponding stochastic differential equation (SDE) is

$$d\mathbf{x}(t) = \mathrm{diag}\left(\mathbf{x}(t)\right)\left(\mathbf{a} + \mathbf{B}\mathbf{x}(t)\right)dt + \boldsymbol{\sigma}\mathbf{x}(t)d\mathbf{w}(t)$$

where $\mathbf{a}, \mathbf{x}, \mathbf{w} \in \mathbb{R}^n, \mathbf{B}, \sigma \in \mathbb{R}^{n \times n}$ with the noisy intensity matrix

$$\sigma = \left(\sigma_{ij}\right)_{n \times n}: \sigma_{ii} > 0 \text{ for } 1 \le i \le n, \sigma_{ij} \ge 0 \text{ for } i \ne j$$

The nonnegative solution may explode in a finite time, since the coefficients do not satisfy the linear growth sufficient condition, though they are locally Lipschitz continuous.

3. Stochastic delay L–V system (continued 1)

- **Delay stochastic LV model:** let the deterministic delay LV system be in the matrix form

$$\dot{\mathbf{x}}(t) = \text{diag}\left(\mathbf{x}(t)\right)\left(\mathbf{b} + \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau)\right), \mathbf{x} \in \mathbb{R}^n$$

and suppose a noisy environment, where the intrinsic growth rates b's are stochastically perturbed, such that

$$b_{ij} \to b_{ij} + \sigma_{ii}(x_j - \bar{x}_j)\dot{w}(t)$$

Then, the corresponding delay SDE is

$$d\mathbf{x}(t) = \text{diag}\left(\mathbf{x}(t)\right)\left(\mathbf{A}(\mathbf{x}(t) - \bar{\mathbf{x}}) + \mathbf{B}(\mathbf{x}(t-\tau) - \bar{\mathbf{x}})\right)dt$$
$$+\boldsymbol{\sigma}(\mathbf{x}(t-\tau) - \bar{\mathbf{x}})\mathrm{d}\mathbf{B}(t),\ \mathbf{x} \in \mathbb{R}^3,\ \boldsymbol{\sigma} = \text{diag}\left(\sigma_{11}, \sigma_{22}, \sigma_{33}\right)$$

3. Stochastic delay L–V system (continued 2)

- **Application:** Stochastic delay LV food chain with delayed interactions for three species.

  The system of SDEs is

$$d\mathbf{x}(t) = \mathrm{diag}\left(\mathbf{x}(t)\right)\left(\mathbf{A}(\mathbf{x}(t) - \bar{\mathbf{x}}) + \mathbf{B}(\mathbf{x}(t-\tau) - \bar{\mathbf{x}})\right)dt$$

$$+\sigma(\mathbf{x}(t-\tau) - \bar{\mathbf{x}})\mathrm{d}\mathbf{B}(t),\ \mathbf{x}\in\mathbb{R}^3,\ \sigma = \mathrm{diag}\left(\sigma_{11,}\sigma_{22},\sigma_{33}\right)$$

  Mao *et al.* (2005) prove that the SS equilibrium is globally asymptotically stable with probability one, if the two folowing conditions are satisfied:

3. Stochastic delay L-V system (continued 2)

- Letting $\hat{c} = a_{11}^{-2} + a_{22}^{-2} + a_{33}^{-2}$, we have the conditions:

$$(i)\ \hat{c}\Big((a_{12}^2 + a_{32}^2) \vee (a_{21}^2 + a_{23}^2)\Big) \leq 1$$

$$(ii)\ \sigma_{ii}^2 \leq \frac{a_{ii}}{\bar{x}_i}\Big(1 - \hat{c}\Big((a_{12}^2 + a_{32}^2) \vee (a_{21}^2 + a_{23}^2)\Big)\Big), \qquad 1 \leq i \leq n$$

- Condition *(i)* garantees that the SS equilibrium of the deterministic system is globally asymptotically stable.
- Condition *(ii)* gives the upper bound for the noise so that the equilibrium of the SDE is still globally asymptotically stable with probability one.

THANK YOU FOR ATTENTION